

(IPS) INNER PRODUCT SPACES

7.1A-1

Inner product spaces are simply vector spaces over the field  $F$  of real or complex numbers and with an Inner Product defined on them.

Def.

Let  $V$  be the vector space over the field  $F$  of real or complex numbers.

A mapping (function)  $\langle \cdot, \cdot \rangle : V \times V \rightarrow F$  is said to be an INNER PRODUCT <sup>(IP)</sup> on  $V$  if the following conditions are satisfied:

- i)  $\langle v, v \rangle \geq 0$   
 $\& \langle v, v \rangle = 0$  iff  $v = 0 \forall v \in V$
- ii)  $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle} \quad \forall v_1, v_2 \in V$   
 where  $\overline{\langle v_2, v_1 \rangle}$  is complex conjugate of  $\langle v_2, v_1 \rangle$
- iii)  $\langle av_1 + bv_2, v_3 \rangle = a\langle v_1, v_3 \rangle + b\langle v_2, v_3 \rangle \quad \forall v_1, v_2, v_3 \in V, a, b \in F$

The pair  $(V, \langle \cdot, \cdot \rangle)$  is called Inner Product Space <sup>(IPS)</sup> where  $V$  is a vector space over the field  $F$  of real or complex numbers and  $\langle \cdot, \cdot \rangle$  is an inner product on  $V$ .

Note If  $F$  is taken as the field of real numbers

then condition (ii) becomes  $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$

$\because$  If  $z$  is real then  $\bar{z} = z$   
 $\therefore \langle v_2, v_1 \rangle = \overline{\langle v_2, v_1 \rangle}$   
 only in real numbers.

We shall consider inner product space over  $\mathbb{R}$  only  
 using cond ii  
 So condition (iii) becomes

$$\langle v_3, av_1 + bv_2 \rangle = \langle av_1 + bv_2, v_3 \rangle \quad \text{using cond (ii)}$$

$$= a\langle v_1, v_3 \rangle + b\langle v_2, v_3 \rangle \quad \text{by (ii)}$$

$$\langle v_3, av_1 + bv_2 \rangle = a\langle v_3, v_1 \rangle + b\langle v_3, v_2 \rangle \quad \text{by (iii)}$$

Q

Example 1 Let  $u, v \in \mathbb{R}^n$  where  $u = (u_1, u_2, \dots, u_n)$ ,  
 $v = (v_1, v_2, \dots, v_n)$

then the dot product  $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$   
 is an inner product on  $\mathbb{R}^n$ , verify.

Sol We show that  $\langle u, v \rangle$  satisfies the three conditions.

$$C_1: \langle u, u \rangle = u_1 u_1 + u_2 u_2 + \dots + u_n u_n \\ = u_1^2 + u_2^2 + \dots + u_n^2 > 0$$

$\therefore$  sum of squares

$$\text{Let } \langle u, u \rangle = 0$$

$$\Rightarrow u_1^2 + u_2^2 + \dots + u_n^2 = 0$$

$$\Rightarrow \text{each } u_i^2 = 0, \quad i = 1, 2, \dots, n$$

$$\Rightarrow \text{each } u_i = 0$$

$$\text{So } u = (0, 0, \dots, 0) = 0$$

$$\text{Hence } \langle u, u \rangle \geq 0$$

$$C_2: \langle u, v \rangle = u_1 v_1 + u_2 v_2 + \dots + u_n v_n \\ = v_1 u_1 + v_2 u_2 + \dots + v_n u_n \\ \langle u, v \rangle = \langle v, u \rangle$$

$u_i, v_i \in \mathbb{R}$

$$C_3: \langle au + bv, w \rangle$$

$$w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$$

$$(au + bv) = a(u_1, u_2, \dots, u_n) + b(v_1, v_2, \dots, v_n)$$

$$= au_1 + bv_1, au_2 + bv_2, \dots, au_n + bv_n$$

$$= (au_1 + bv_1)w_1 + (au_2 + bv_2)w_2 + \dots + (au_n + bv_n)w_n \quad (\text{as defind})$$

$$= au_1 w_1 + bv_1 w_1 + au_2 w_2 + bv_2 w_2 + \dots + au_n w_n + bv_n w_n$$

$$= a(u_1 w_1 + u_2 w_2 + \dots + u_n w_n) + b(v_1 w_1 + v_2 w_2 + \dots + v_n w_n)$$

$$= a \langle u, w \rangle + b \langle v, w \rangle$$

Hence  $\langle u, v \rangle$  is an Inner Product on  $\mathbb{R}^n$ .

Note  $(\mathbb{R}^n, \langle u, v \rangle)$  is called Euclidean Inner Product Space on  $\mathbb{R}^n$

where  $\langle u, v \rangle = \text{dot product}$

$$= u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Example 2 Let  $V$  be the vector space of all  $n \times 1$  matrices over  $\mathbb{R}$

$$c_1, c_2 \in V \text{ where } c_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\text{and } c_2 = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

(Such matrices are called Column Vectors)

$$c_1^t = [x_1, x_2, \dots, x_n]$$

$$c_2^t = [y_1, y_2, \dots, y_n]$$

Show that  $\langle c_1, c_2 \rangle$  is IP, where  $\langle c_1, c_2 \rangle = \det(c_1^t, c_2^t) = |c_1^t, c_2^t|$

Sol

$$c_1^t c_2 = [x_1, x_2, \dots, x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$c_1^t c_2 = (x_1 y_1 + x_2 y_2 + \dots + x_n y_n)$$

$$|c_1^t c_2| = |x_1 y_1 + x_2 y_2 + \dots + x_n y_n| = x_1 y_1 + x_2 y_2 + \dots + x_n y_n \quad \text{--- (1)}$$

$1 \times n$   
 $n \times 1$   
 $= [x]$   
 $= \text{determinant}$

$$c_1: \langle c_1, c_1 \rangle = |c_1^t c_1|$$

$$= x_1 x_1 + x_2 x_2 + \dots + x_n x_n = x_1^2 + x_2^2 + \dots + x_n^2 > 0$$

$$\text{Let } \langle c_1, c_1 \rangle = 0$$

$$\Leftrightarrow x_1^2 + x_2^2 + \dots + x_n^2 = 0$$

$$\Leftrightarrow \text{each } x_i = 0 \quad i = 1, 2, \dots, n.$$

$$\Leftrightarrow c_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \quad \therefore \langle c_1, c_1 \rangle = 0 \Rightarrow c_1 = 0$$

$$\begin{aligned} c_2: \langle c_2, c_2 \rangle &= |c_2^t c_2| = x_1 y_1 + x_2 y_2 + \dots + x_n y_n \\ &= y_1 x_1 + y_2 x_2 + \dots + y_n x_n \\ &= |c_2^t c_1| \\ &= \langle c_2, c_1 \rangle \end{aligned}$$

$$c_3: \langle ac_1 + bc_2, c_3 \rangle = |(ac_1 + bc_2)^t c_3|$$

$$\text{Let } a, b \in \mathbb{R} \text{ and } c_3 = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$ac_1 + bc_2 = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ \vdots \\ ax_n + by_n \end{bmatrix}$$

$$= |(ax_1 + by_1)z_1 + (ax_2 + by_2)z_2 + \dots + (ax_n + by_n)z_n|$$

$$= (ax_1 + by_1)z_1 + (ax_2 + by_2)z_2 + \dots + (ax_n + by_n)z_n$$

$$= a(x_1 z_1 + x_2 z_2 + \dots + x_n z_n) + b(y_1 z_1 + y_2 z_2 + \dots + y_n z_n)$$

$$= a |c_1^t c_3| + b |c_2^t c_3|$$

$$= a \langle c_1, c_3 \rangle + b \langle c_2, c_3 \rangle$$

Hence  $\langle c_1, c_2 \rangle$  is an IP on  $V$

Example 3 Let  $u, v \in \mathbb{R}^2$ ,  $u = (x_1, x_2)$   $v = (y_1, y_2)$

Then show that  $\langle u, v \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3x_2 y_2$  is an IP on  $\mathbb{R}^2$

Sol (i)  $\langle u, u \rangle = x_1 x_1 - x_1 x_2 - x_2 x_1 + 3x_2 x_2$

$$= x_1^2 - 2x_1 x_2 + 3x_2^2$$

$$= x_1^2 - 2x_1 x_2 + x_2^2 + 2x_2^2$$

$$= (x_1 - x_2)^2 + 2x_2^2 \geq 0$$

Let  $\langle u, u \rangle = 0$

$$\Leftrightarrow (x_1 - x_2)^2 + 2x_2^2 = 0$$

$$\Leftrightarrow (x_1 - x_2)^2 + 2x_2^2 = 0$$

$$\Leftrightarrow (x_1 - x_2) = 0 \text{ \& } \sqrt{2}x_2 = 0$$

$$\Leftrightarrow x_1 = x_2 \text{ \& } x_2 = 0$$

$$\Leftrightarrow x_1 = 0, \text{ \& } x_2 = 0$$

$$\Leftrightarrow u = 0$$

(ii)  $\langle u, v \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3x_2 y_2$

$$= y_1 x_1 - y_2 x_1 - y_1 x_2 + 3y_2 x_2$$

$$= y_1 x_1 - y_1 x_2 - y_2 x_1 + 3y_2 x_2$$

$$= \langle v, u \rangle \quad \forall u, v \in \mathbb{R}^2$$

(iii)  $\langle au + bv, w \rangle = \langle (ax_1 + by_1, ax_2 + by_2), (z_1, z_2) \rangle$

$$= (ax_1 + by_1)z_1 - (ax_1 + by_1)z_2 - (ax_2 + by_2)z_1 + 3(ax_2 + by_2)z_2$$

$$= ax_1 z_1 + by_1 z_1 - ax_1 z_2 - by_1 z_2 - ax_2 z_1 - by_2 z_1 + 3ax_2 z_2 + 3by_2 z_2$$

$$= a(x_1 z_1 - x_1 z_2 - x_2 z_1 + 3x_2 z_2) + b(y_1 z_1 - y_1 z_2 - y_2 z_1 + 3y_2 z_2)$$

$$= a \langle u, w \rangle + b \langle v, w \rangle$$

Thus all the three conditions are satisfied

So  $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$  is an IPS.

Let  $w = (z_1, z_2)$

$$= au + bv$$

$$= a(x_1, x_2) + b(y_1, y_2)$$

$$= (ax_1 + by_1, ax_2 + by_2)$$

## Norm (or Length) of a vector.

Let  $V$  be an IPS and  $v \in V$ , then the real number  $\sqrt{\langle v, v \rangle}$  is called the norm of  $v$ , it is denoted by  $\|v\|$ .

$$\therefore \|v\| = \sqrt{\langle v, v \rangle} \quad \& \quad \|v\|^2 = \langle v, v \rangle$$

$$\text{If } \|v\| = 1$$

$$\text{i.e. } \sqrt{\langle v, v \rangle} = 1$$

i.e.  $\langle v, v \rangle = 1$  then  $v$  is called Unit vector  
or Normalized Vector.

Any non-zero vector  $u \in V$  can be normalized by multiplying it with  $\frac{1}{\|u\|}$

$\therefore \frac{u}{\|u\|}$  is a unit vector i.e. vector ' $u$ ' has been Normalized.

Example 5 Find the norm of  $v = (3, 4) \in \mathbb{R}^2$  with respect to the Euclidean inner product and the inner product defined in Example 3.  
Sol Euclidean inner product on  $\mathbb{R}^n$  is defined as

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$\text{where } u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n \\ v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$$

$$\|v\| = \sqrt{\langle v, v \rangle}$$

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} \quad \text{on } \mathbb{R}^n$$

$$\|v\| = \sqrt{v_1^2 + v_2^2} \quad \text{on } \mathbb{R}^2$$

$$= \sqrt{9+16}$$

$$= 5$$

$$v = (3, 4) \in \mathbb{R}^2$$

$$\therefore v_1 = 3, v_2 = 4$$

$$\text{ii) } \langle u, v \rangle = \langle (u_1, u_2), (v_1, v_2) \rangle$$

$$= u_1 v_1 - u_1 v_2 - u_2 v_1 + 3 u_2 v_2$$

$$\|v\| = \sqrt{\langle v, v \rangle}$$

$$= \sqrt{v_1 v_1 - v_1 v_2 - v_2 v_1 + 3 v_2 v_2}$$

$$= \sqrt{3 \cdot 3 - 3 \cdot 4 - 4 \cdot 3 + 3 \cdot 4 \cdot 4} = \sqrt{9 - 12 - 12 + 48} = \sqrt{33} \text{ Ans.}$$

as defined in Example 3

Example 4 Let  $V$  be the vector space of all real-valued continuous functions on the interval  $a \leq t \leq b$ , then for  $f, g \in V$

$$\langle f, g \rangle = \int_a^b f(t)g(t) dt \quad \text{is an inner product on } V.$$

Sol (i)  $\langle f, f \rangle = \int_a^b f(t)f(t) dt$

$$= \int_a^b f^2(t) dt \geq 0 \quad \because \text{the definite integral gives the area bounded by the curve, which is always +ve.}$$

$$\text{Let } \langle f, f \rangle = 0$$

$$\Leftrightarrow \int_a^b f^2(t) dt = 0$$

$$\Leftrightarrow f^2(t) = 0$$

$$\Leftrightarrow f(t) = 0$$

$$\begin{aligned} \text{(ii) } \langle f, g \rangle &= \int_a^b f(t)g(t) dt \\ &= \int_a^b g(t)f(t) dt \\ &= \langle g, f \rangle \end{aligned}$$

$$\begin{aligned} \text{(iii) } \langle a_1 f + b_1 g, h \rangle &= \int_a^b (a_1 f(t) + b_1 g(t)) h(t) dt \\ &= \int_a^b (a_1 f(t) h(t) + b_1 g(t) h(t)) dt \\ &= a_1 \int_a^b f(t) h(t) dt + b_1 \int_a^b g(t) h(t) dt \end{aligned}$$

$$= a_1 \langle f, h \rangle + b_1 \langle g, h \rangle \quad \begin{array}{l} \forall a_1, b_1 \in \mathbb{R} \\ \forall f, g, h \in V \end{array}$$

Thus all the three conditions of an I.P are satisfied

So  $(V, \langle \cdot, \cdot \rangle)$  is an IPS

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Sample 6

Normalize  $v = (1, 2, 1) \in \mathbb{R}^3$  w.r.t. Euclidean I.P

$$\begin{aligned} \text{Sol } \|v\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{1^2 + 2^2 + 1^2} \\ &= \sqrt{6} \end{aligned}$$

$$\therefore \|v\| = \sqrt{\langle v, v \rangle}$$

$$\text{Normalized Vector} = \frac{v}{\|v\|} = \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$$

The Cauchy-Schwarz Inequality

Let  $u, v$  be the elements of an inner product space  $V$  or  $\mathbb{R}$

$$\text{then } |\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

Proof If  $v=0$  then

$$\text{LHS } \langle u, v \rangle = \langle u, 0 \rangle \quad \therefore v=0$$

$$= \langle u, 0 \cdot w \rangle \quad \forall w \in V$$

$$= 0 \langle u, w \rangle \quad \text{by cond (iii)}$$

$$= 0$$

$$\text{RHS } \|u\| \|v\| = \|u\| \|0\|$$

$$= 0$$

$$\therefore \text{LHS} = \text{RHS.}$$

If  $v \neq 0$  then for all real  $t \in \mathbb{R}$

$$0 \leq \|u - tv\|^2$$

$$a=1, b=-t$$

$$= \langle u - tv, u - tv \rangle$$

$$= \langle u, u - tv \rangle - t \langle v, u - tv \rangle \quad \text{by cond (ii)}$$

$$= \langle u, u \rangle - t \langle u, v \rangle - t \langle v, u \rangle + t^2 \langle v, v \rangle \quad \text{by cond (ii)}$$

$$= \|u\|^2 - 2t \langle u, v \rangle + t^2 \|v\|^2$$

$$\text{Let } t = \frac{\langle u, v \rangle}{\langle v, v \rangle}, \Rightarrow t^2 = \frac{|\langle u, v \rangle|^2}{\|v\|^4}$$

$$\begin{aligned} t \cdot \bar{t} &= |t|^2 \\ \text{Fixed } t^2 &= |t|^2 \\ \therefore \langle u, v \rangle \langle u, v \rangle &= |\langle u, v \rangle|^2 \end{aligned}$$

$$0 \leq \|u\|^2 - 2 \frac{\langle u, v \rangle \langle u, v \rangle}{\langle v, v \rangle} + \frac{|\langle u, v \rangle|^2}{\|v\|^4} \|v\|^2$$

$$0 \leq \|u\|^2 - 2 \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \frac{|\langle u, v \rangle|^2}{\|v\|^2}$$

$$\text{by LEM } 0 \leq \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 \Rightarrow |\langle u, v \rangle| \leq \|u\| \|v\|$$

Taking Square Root

proved.

Q.

Theorem The norm in an inner product space  $V$  satisfies the following axioms

$$(i) \|v\| \geq 0 \quad \text{and} \quad \|v\| = 0 \iff v = 0, v \in V$$

$$(ii) \|kv\| = |k| \|v\| \quad \text{for all } v \in V \text{ and } k \in \mathbb{R}$$

$$(iii) \|u+v\| \leq \|u\| + \|v\| \quad \text{for all } u, v \in V$$

Proof (i)  $\|v\| = \sqrt{\langle v, v \rangle}$

$$\geq 0 \quad \because \langle v, v \rangle \geq 0 \text{ by condition (i)}$$

$$\therefore \|v\| \geq 0$$

$$\text{Further } \|v\| = 0 \iff \sqrt{\langle v, v \rangle} = 0$$

$$\iff \langle v, v \rangle = 0$$

$$\iff v = 0$$

$$(ii) \|kv\|^2 = \langle kv, kv \rangle \quad k \in \mathbb{R}$$

$$= k \langle v, kv \rangle$$

$$= k^2 \langle v, v \rangle$$

$$= k^2 \|v\|^2 \quad \because \sqrt{\langle v, v \rangle} = \|v\|$$

$$= |k|^2 \|v\|^2 \quad \because k \in \mathbb{R}$$

$$(iii) \|u+v\|^2$$

$$= \langle u+v, u+v \rangle$$

$$= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle$$

$$z + \bar{z} = 2\operatorname{Re} z$$

$$= \|u\|^2 + 2\langle u, v \rangle + \|v\|^2$$

$$\because \text{condition (i)} \\ \therefore \langle u, v \rangle = \langle v, u \rangle$$

$$\leq \|u\|^2 + 2|\langle u, v \rangle| + \|v\|^2$$

$$(\operatorname{Re} z \leq |z|)$$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad \text{By Cauchy-Schwarz}$$

$$\langle u, v \rangle \leq |\langle u, v \rangle|$$

$$= (\|u\| + \|v\|)^2$$

$$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$$

$$\|u+v\| \leq \|u\| + \|v\|$$

ORTHOGONALITY

Let  $\theta$  be the angle between two vectors  $u, v \in V(\text{IPS})$ , then

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|} \quad 0 \leq \theta \leq \pi$$

$\therefore$  By Cauchy Schwarz

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

$$\frac{|\langle u, v \rangle|}{\|u\| \|v\|} \leq \frac{\|u\| \|v\|}{\|u\| \|v\|}$$

$$\frac{|\langle u, v \rangle|}{\|u\| \|v\|} \leq 1$$

$$\text{If } \theta = 90$$

$$\cos(90) = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

$$0 = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

$$0 = \langle u, v \rangle \quad \therefore$$

$\therefore u, v$  are orthogonal if  $\langle u, v \rangle = 0$   
( $u \perp v$ , read  $u$  is orthogonal to  $v$ )

Note

$$\text{If } u \perp v \Rightarrow \langle u, v \rangle = 0$$

$$\text{If } v \perp u \Rightarrow \langle v, u \rangle = 0$$

$$\therefore \langle u, v \rangle = \langle v, u \rangle$$

Hence relation of orthogonality is symmetric.

2) The vector '0' is orthogonal to every  $v \in V$

$$\therefore \langle 0, v \rangle = \langle 0v, v \rangle = 0 \cdot \langle v, v \rangle = 0 \text{ by cond iii}$$

3) If  $u$  is orthogonal to itself then  $u = 0$

$$\therefore \langle u, u \rangle = 0 \quad \text{by def of orthogonality}$$

$$\|u\|^2 = 0 \Rightarrow \|u\| = 0 \Rightarrow u = 0$$

Example 7 Show that  $x \perp y$  where  $x, y \in \mathbb{R}^3$

$$x = (1, -1, 2) \quad y = (-1, 1, 1)$$

$$\begin{aligned} \text{Sol } \langle x, y \rangle &= (1)(-1) + (-1)(1) + (2)(1) \\ &= -1 - 1 + 2 \end{aligned}$$

$$\langle x, y \rangle = 0 \quad \text{So } x \perp y$$

Show that  $x \perp y$  where  $x, y \in \mathbb{R}^4$

$$x = (1, -1, 1, -1) \quad y = (-1, 2, 2, -1)$$

$$\langle x, y \rangle = (1)(-1) + (-1)(2) + (1)(2) + (-1)(-1)$$

$$= -1 - 2 + 2 + 1$$

$$= 0 \quad \text{So } x \perp y$$

Example 8 If  $u$  is orthogonal to  $v$  then every scalar multiple of  $u$  is also orthogonal to  $v$ . ( $Ku$  is multiple of  $u$ )

Sol If  $u \perp v \Rightarrow \langle u, v \rangle = 0$

$$\text{then } \langle Ku, v \rangle = K \langle u, v \rangle \quad \text{by cond iii} \\ = K(0)$$

$$\langle Ku, v \rangle = 0$$

Hence  $Ku \perp v$

Example 9 Find a unit vector orthogonal to both  $(1, 1, 2)$  and  $(0, 1, 3)$  in  $\mathbb{R}^3$ .

Sol Let  $(x, y, z) \in \mathbb{R}^3$  be a vector orthogonal to given vectors

$$\therefore \langle (x, y, z), (1, 1, 2) \rangle = 0 \quad \therefore (x, y, z) \perp (1, 1, 2)$$

$$x(1) + y(1) + z(2) = 0 \quad \text{--- (I)}$$

$$\text{Also } \langle (x, y, z), (0, 1, 3) \rangle = 0$$

$$\therefore (x, y, z) \perp (0, 1, 3)$$

$$x(0) + y(1) + z(3) = 0 \quad \text{--- (II)}$$

$$\text{using (II) in (I)} \quad y = -3z$$

$$x - 3z + 2z = 0$$

$$x - z = 0$$

$$x = z$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -3z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$\Rightarrow (1, -3, 1)$  is  $\perp$  to both  $u$  &  $v$

$$\|(1, -3, 1)\| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$$

$$\text{Unit Vector} = \left( \frac{1}{\sqrt{11}}, \frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right)$$

2nd Method

$$w = \text{vector } \perp u + v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$$e_1 = (1, 0, 0) = i$$

$$e_2 = (0, 1, 0) = j$$

$$e_3 = (0, 0, 1) = k$$

$$w = e_1(3-2) - e_2(3-0) + e_3(1-0)$$

$$= e_1 - 3e_2 + e_3 = 1, -3, 1$$

$$\text{Unit Vector} = \frac{w}{\|w\|} = \frac{1}{\sqrt{11}} (e_1 - 3e_2 + e_3) = \frac{1}{\sqrt{11}} e_1 - \frac{3}{\sqrt{11}} e_2 + \frac{1}{\sqrt{11}} e_3$$

Orthogonal Complement..

Let  $W$  be a subset of an Inner Product Space  $V$  over  $R$ . The orthogonal complement of  $W$  denoted by  $W^\perp$  and read as ' $W$  perp', consists of those vectors in  $V$  which are orthogonal to every  $w \in W$ . Thus

$$W^\perp = \{v \in V : \langle v, w \rangle = 0 \quad \forall w \in W\}$$

Prove that  $W^\perp$  is subspace of  $V$ .

Let  $u, v \in W^\perp$  then for  $w \in W$  and  $a, b \in R$

$$\langle u, w \rangle = 0 \quad \because u \in W^\perp$$

$$\langle v, w \rangle = 0$$

$$\begin{aligned} \text{Since } \langle au + bv, w \rangle &= a \langle u, w \rangle + b \langle v, w \rangle \\ &= a \cdot 0 + b \cdot 0 \\ &= 0 \end{aligned}$$

So  $au + bv \in W^\perp$ . Thus,  $W^\perp$  is a subspace of  $V$ .

Orthogonal System

A set  $S$  of vectors in an I.P. Space  $V$  over  $R$  is said to be an orthogonal system if its distinct vectors are orthogonal, i.e. if

$$\langle u_i, u_j \rangle = 0 \quad \forall u_i, u_j \in S \quad i \neq j$$
Orthonormal System

A set  $S$  of vectors in an I.P. Space  $V$  over  $R$  is said to be an orthonormal system if

$$\langle u_i, u_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example 10Show that the system  $\{u_1 = (1, -1, 1, -1)$ 

$$u_2 = (3, 1, -1, 1)$$

$$u_3 = (0, 2, 1, -1)$$

$$u_4 = (0, 0, 1, 1)\}$$
 is an orthonormal system in  $\mathbb{R}^4$ .

sol

$$\langle u_1, u_2 \rangle = 1 \cdot 3 + (-1)(1) + 1(-1) + (-1)(1)$$

$$= 3 - 1 - 1 - 1$$

$$= \boxed{0}$$

$$\langle u_1, u_3 \rangle = 1 \cdot 0 + (-1)(2) + 1(1) + (-1)(-1)$$

$$= 0 - 2 + 1 - 1$$

$$= \boxed{0}$$

$$\langle u_1, u_4 \rangle = 1 \cdot 0 + (-1)(0) + 1(1) + (-1)(1)$$

$$= 0 + 0 + 1 - 1$$

$$= \boxed{0}$$

$$\langle u_2, u_3 \rangle = (3)(0) + (1)(2) + (-1)(1) + 1(-1)$$

$$= 0 + 2 - 1 - 1$$

$$= \boxed{0}$$

$$\langle u_2, u_4 \rangle = (3)(0) + 1(0) + (-1)(1) + 1(1)$$

$$= 0 + 0 - 1 + 1$$

$$= \boxed{0}$$

$$\langle u_3, u_4 \rangle = (0)(0) + (-1)(0) + 1(1) + (-1)(1)$$

$$= 0 + 0 + 1 - 1$$

$$= \boxed{0}$$

$$\langle u_1, u_1 \rangle = 1(1) + (-1)(-1) + 1(1) + (-1)(-1)$$

$$= 1 + 1 + 1 + 1$$

$$= 4 \neq 1$$

$$\langle u_2, u_2 \rangle = (3)(3) + (1)(1) + (-1)(-1) + 1(1)$$

$$= 9 + 1 + 1 + 1$$

$$= 12 \neq 1$$

So this is orthogonal system not orthonormal system.

Def

Example 11 Let  $V$  be the vector space of real-valued continuous fns on the interval  $-\pi \leq t \leq \pi$  with inner product defined by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \cdot g(t) dt \quad \forall f, g \in V$$

then  $\{1, \cos t, \cos 2t, \dots, \sin t, \sin 2t, \dots\}$  is an orthogonal system in  $V$ .

Sol  $\langle 1, \cos mt \rangle = \int_{-\pi}^{\pi} 1 \cdot \cos mt dt = \left| \frac{\sin mt}{m} \right|_{-\pi}^{\pi} = \frac{1}{m} [\sin(m\pi) - \sin(-m\pi)]$

$$= \frac{1}{m} (0 - 0) = \boxed{0}$$

$$\langle 1, \sin mt \rangle = \int_{-\pi}^{\pi} 1 \cdot \sin mt dt = \left| -\frac{\cos mt}{m} \right|_{-\pi}^{\pi} = -\frac{1}{m} [\cos m\pi + \cos(-m\pi)]$$

$$= -\frac{1}{m} (\cos m\pi + \cos m\pi)$$

$$= \boxed{0}$$

when  $m \neq n$ 

$$\langle \cos mt, \sin nt \rangle = \int_{-\pi}^{\pi} \cos(mt) \sin(nt) dt = \frac{1}{2} \int_{-\pi}^{\pi} 2 \cos(mt) \sin(nt) dt$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \{ \sin(m+n)t - \sin(n-m)t \} dt$$

$$\because 2 \cos \alpha \sin \beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$$

where  $m > n$ 

$$= \frac{1}{2} \left[ -\frac{\cos(m+n)t}{(m+n)} - \left( -\frac{\cos(m-n)t}{m-n} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} \left\{ \left( -\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n} \right) - \left( -\frac{\cos(m+n)(-\pi)}{m+n} + \frac{\cos(m-n)(-\pi)}{m-n} \right) \right\}$$

$$= \frac{1}{2} \left\{ -\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n} + \frac{\cos(m+n)(-\pi)}{m+n} - \frac{\cos(m-n)(-\pi)}{m-n} \right\}$$

$$= \boxed{0}$$

$$\because \cos(-\pi) = \cos \pi$$

when  $m = n$ 

$$\langle \cos mt, \sin mt \rangle = \int_{-\pi}^{\pi} \cos mt \sin mt dt$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} 2 \cos mt \sin mt dt = \frac{1}{2} \int_{-\pi}^{\pi} \sin 2mt dt \quad \because 2 \cos \theta \sin \theta = \sin 2\theta$$

$$= \frac{1}{2} \left| -\frac{\cos 2mt}{2m} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{4m} (\cos 2m\pi + \cos 2m(-\pi))$$

$$= \boxed{0}$$

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when  $m \neq n$ 

$$\langle \cos mt, \cos nt \rangle = \int_{-\pi}^{\pi} \cos mt \cos nt \, dt$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} 2 \cos mt \cos nt \, dt$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \cos(m+n)t + \cos(m-n)t \, dt$$

$$= \frac{1}{2} \left[ \frac{\sin(m+n)t}{m+n} + \frac{\sin(m-n)t}{m-n} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} \left[ \frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)(-\pi)}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n} \right]$$

$$= \boxed{0}$$

$$\because \sin \pi = 0 = \sin(-\pi)$$

when  $m \neq n$ 

$$\langle \sin mt, \sin nt \rangle = \int_{-\pi}^{\pi} \sin mt \sin nt \, dt$$

$$= -\frac{1}{2} \int_{-\pi}^{\pi} 2 \sin mt \sin nt \, dt$$

$$= -\frac{1}{2} \int_{-\pi}^{\pi} \cos(m+n)t - \cos(m-n)t \, dt$$

$$= -\frac{1}{2} \left[ \frac{\sin(m+n)t}{m+n} - \frac{\sin(m-n)t}{m-n} \right]_{-\pi}^{\pi}$$

$$= -\frac{1}{2} \left[ \frac{\sin(m+n)\pi}{m+n} - \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)(-\pi)}{m+n} + \frac{\sin(m-n)(-\pi)}{m-n} \right]$$

$$= \boxed{0}$$

Thus the set  $\{1, \cos t, \cos 2t, \dots, \sin t, \sin 2t, \dots\}$  is an orthogonal system in  $V$ .

Ex: when  $m=n$ 

$$\langle \cos t, \cos t \rangle = \int_{-\pi}^{\pi} \cos^2 t \, dt = \int_{-\pi}^{\pi} \left( \frac{1 + \cos 2t}{2} \right) dt = \frac{1}{2} \left[ t + \frac{\sin 2t}{2} \right]_{-\pi}^{\pi} =$$

$$= \frac{1}{2} \left[ \pi + \frac{\sin 2\pi}{2} - (-\pi) - \frac{\sin 2(-\pi)}{2} \right] = \frac{1}{2} (\pi + \pi) = \boxed{\pi}$$

when  $m=n$ 

$$\langle \sin t, \sin t \rangle = \int_{-\pi}^{\pi} \sin^2 t \, dt = \int_{-\pi}^{\pi} \left( \frac{1 - \cos 2t}{2} \right) dt = \frac{1}{2} \left[ t - \frac{\sin 2t}{2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} \left[ \pi - \frac{\sin 2\pi}{2} - (-\pi) + \frac{\sin 2(-\pi)}{2} \right] = \frac{1}{2} (\pi + \pi) = \boxed{\pi}$$

# The Gram-Schmidt Process

$V$  is an I.P. Space over  $\mathbb{R}$ .

$\{v_1, v_2, \dots, v_n\}$  is a basis of  $V(\mathbb{R})$ .

An orthonormal basis  $\{u_1, u_2, \dots, u_n\}$  of  $V$  can be constructed as,

Step 1 Let  $u_1 = \frac{v_1}{\|v_1\|}$

$$\|u_1\| = \|v_1\| \cdot \frac{1}{\|v_1\|} = 1$$

$u_1$  is written in the linear combination of  $v_1$  &  $\|v_1\|$  where  $\|v_1\|$  is scalar. Similarly  $v_1 = u_1 \|v_1\|$  So  $\{u_1\} = \{v_1\}$

Step 2

Let  $u_2 = \frac{w_2}{\|w_2\|}$

where  $w_2 = v_2 - \langle v_2, u_1 \rangle u_1$

$$\langle u_1, u_2 \rangle = \langle u_1, \frac{w_2}{\|w_2\|} \rangle$$

$$= \frac{1}{\|w_2\|} \langle u_1, w_2 \rangle$$

$$= \frac{1}{\|w_2\|} \langle u_1, v_2 - \langle v_2, u_1 \rangle u_1 \rangle$$

$$= \frac{1}{\|w_2\|} \left[ \langle u_1, v_2 \rangle - \langle v_2, u_1 \rangle \langle u_1, u_1 \rangle \right]$$

by cond iii

$$= \frac{1}{\|w_2\|} \left[ \langle u_1, v_2 \rangle - \langle u_1, v_2 \rangle \cdot 1 \right]$$

$$= \|u_1\| = \sqrt{\langle u_1, u_1 \rangle}$$

$$1 = \sqrt{\langle u_1, u_1 \rangle}$$

$$\text{sqing } 1 = \langle u_1, u_1 \rangle$$

$$\langle u_1, u_2 \rangle = 0$$

$\neq$

$$\|u_2\| = \frac{\|w_2\|}{\|w_2\|} = 1$$

Hence  $\{u_1, u_2\}$  is orthonormal

Hence  $\{u_1, u_2\}$  is O.N.S.

Step 3 Let  $u_3 = \frac{w_3}{\|w_3\|}$

where  $w_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2$

Similarly  $\{u_1, u_2, u_3\}$  is orthonormal.

and so on, then  $u_n = \frac{w_n}{\|w_n\|}$

where  $w_n = v_n - \langle v_n, u_1 \rangle u_1 - \langle v_n, u_2 \rangle u_2 - \dots$

$$- \dots - \langle v_n, u_{n-1} \rangle u_{n-1}$$

$\therefore$  The set  $\{u_1, u_2, \dots, u_n\}$  is orthonormal

$\therefore$  We know orthonormal is linearly independent and

$$\text{Span} \{u_1, u_2, \dots, u_n\} = \text{Span} \{v_1, v_2, \dots, v_n\}$$

Hence  $\{u_1, u_2, \dots, u_n\}$  is orthonormal basis of  $V$ .

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Theorem Every orthonormal system  $\{u_1, u_2, \dots, u_n\}$  is linearly independent.

Moreover, for all  $v \in V$ , the vector  $w = v - \sum_{k=1}^n \langle v, u_k \rangle u_k$

is orthogonal to each  $u_i$ ,  $1 \leq i \leq n$ . i.e.  $\{u_1, u_2, \dots, u_n\}$

Proof Suppose  $a_1 u_1 + a_2 u_2 + \dots + a_n u_n = 0$

where  $a_i$  are scalars  
 $a_i \in \mathbb{R}$

Taking inner product of both sides with ' $u_i$ '

$$\langle a_1 u_1 + a_2 u_2 + \dots + a_n u_n, u_i \rangle = \langle 0, u_i \rangle$$

$$a_1 \langle u_1, u_i \rangle + a_2 \langle u_2, u_i \rangle + \dots + a_i \langle u_i, u_i \rangle + \dots + a_n \langle u_n, u_i \rangle$$

$$+ a_i \langle u_i, u_i \rangle + a_{i+1} \langle u_{i+1}, u_i \rangle + \dots + a_n \langle u_n, u_i \rangle = 0$$

$\therefore$  Cond iii

$\therefore 0$  is orthogonal to every  $v \in V$   
 $\therefore \langle 0, v \rangle = 0$

$$0 + 0 + \dots + a_i \cdot 1 + 0 + \dots + 0 = 0$$

$$a_i \cdot 1 = 0$$

$$a_i = 0$$

$\therefore 1 \neq 0$   
 $a_i = 0$

$\forall i = 1, 2, \dots, n$

$\left\{ \begin{array}{l} \therefore \{u_1, u_2, \dots, u_n\} \text{ is orthonormal system} \\ \therefore \langle u_i, u_j \rangle = 0, i \neq j \text{ distinct} \\ \langle u_i, u_i \rangle = 1 \quad i=j \end{array} \right.$

$\therefore \{u_1, u_2, \dots, u_n\}$  is linearly independent.

Now to prove  $w$  is orthogonal to each  $u_i$ ,  $1 \leq i \leq n$

Consider  $\langle w, u_i \rangle = \langle v - \sum_{k=1}^n \langle v, u_k \rangle u_k, u_i \rangle$

$$= \langle v, u_i \rangle - \langle \sum_{k=1}^n \langle v, u_k \rangle u_k, u_i \rangle$$

$\therefore$  Cond iii

$$= \langle v, u_i \rangle - \langle \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 + \dots + \langle v, u_n \rangle u_n, u_i \rangle$$

$$= \langle v, u_i \rangle - (\langle v, u_1 \rangle \langle u_1, u_i \rangle + \langle v, u_2 \rangle \langle u_2, u_i \rangle + \dots$$

$$- \langle v, u_{i-1} \rangle \langle u_{i-1}, u_i \rangle - \langle v, u_i \rangle \langle u_i, u_i \rangle - \langle v, u_{i+1} \rangle \langle u_{i+1}, u_i \rangle - \dots - \langle v, u_n \rangle \langle u_n, u_i \rangle$$

$\therefore \langle v, u \rangle \in F = \mathbb{R}$   $\therefore$  by Cond iii

$$- \langle v, u_{i+1} \rangle \langle u_{i+1}, u_i \rangle - \dots - \langle v, u_n \rangle \langle u_n, u_i \rangle$$

$$= \langle v, u_i \rangle - 0 - 0 - \dots - 0 - \langle v, u_i \rangle \cdot 1 - 0 - \dots - 0$$

$\left( \begin{array}{l} \therefore \langle u_i, u_j \rangle = 0 \\ \langle u_i, u_i \rangle = 1 \end{array} \right.$

$$= \langle v, u_i \rangle - \langle v, u_i \rangle$$

$$\langle w, u_i \rangle = \boxed{0}$$

Hence  $w$  is orthogonal to each  $u_i$ ,  $1 \leq i \leq n$

x-----x

Example 12 Show that  $\{(1,1,1), (0,1,1), (0,0,1)\}$  is a basis of  $\mathbb{R}^3$ .

Using Gram-Schmidt orthogonalization process, transform this basis into an orthonormal basis.

Sol Let  $(x, y, z) \in \mathbb{R}^3$

$$\text{Suppose } (x, y, z) = a(1,1,1) + b(0,1,1) + c(0,0,1)$$

$$(x, y, z) = (a, a+b, a+b+c)$$

Linear combination of  
 $a, b, c$  of given set of vectors

$$\therefore \boxed{a = x}$$

$$a+b = y \Rightarrow b = y-a \Rightarrow \boxed{b = y-x}$$

$$a+b+c = z \Rightarrow c = z-a-b \Rightarrow c = z-x-(y-x)$$

$$c = z-x-y+x$$

$$\boxed{c = z-y}$$

$\therefore$  the given set of vectors  $\{(1,1,1), (0,1,1), (0,0,1)\}$  can be written as a linear combination of  $a, b, c$ , as values of  $a, b, c$  exist.

$\therefore$  the set  $\{(1,1,1), (0,1,1), (0,0,1)\}$  is a spanning set for  $\mathbb{R}^3$

Now check Linear Independence of  $\{(1,1,1), (0,1,1), (0,0,1)\}$

$$a(1,1,1) + b(0,1,1) + c(0,0,1) = 0$$

$$(a, a+b, a+b+c) = 0$$

$$\boxed{a = 0}$$

$$a+b = 0 \Rightarrow 0+b = 0 \Rightarrow \boxed{b = 0}$$

$$a+b+c = 0 \Rightarrow 0+0+c = 0 \Rightarrow \boxed{c = 0}$$

Hence given vectors are Lin Independent.

Since the given set of vectors is a Spanning set for  $\mathbb{R}^3$  and is Linearly Independent. Hence

Hence the given set of vectors  $\{(1,1,1), (0,1,1), (0,0,1)\}$  is a basis of  $\mathbb{R}^3$ .

P.T.O

2nd Method

Since the Matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ Rank of } A = 3$$

= No. of vectors = 3

is in Echelon form

So the given three vectors  $(1,1,1), (0,1,1), (0,0,1)$  are Linearly Independent.

by Gram Schmidt Orthogonalization.

Let  $v_1 = (1, 1, 1)$ ,  $v_2 = (0, 1, 1)$ ,  $v_3 = (0, 0, 1)$

Now  $u_1 = \frac{v_1}{\|v_1\|} = \frac{(1, 1, 1)}{\sqrt{1^2+1^2+1^2}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$u_2 = \frac{\omega_2}{\|\omega_2\|}$  where  $\omega_2 = v_2 - \langle v_2, u_1 \rangle u_1$

$$= (0, 1, 1) - \langle (0, 1, 1), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \rangle \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= (0, 1, 1) - \left(0 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= (0, 1, 1) - \frac{2}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= (0, 1, 1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$\omega_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$\|\omega_2\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}}$$

$\therefore u_2 = \frac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}{\sqrt{\frac{2}{3}}}$

$u_2 = \frac{\sqrt{3}}{2} \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$u_3 = \frac{\omega_3}{\|\omega_3\|}$  where  $\omega_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2$

$$= (0, 0, 1) - \langle (0, 0, 1), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \rangle \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) -$$

$$- \langle (0, 0, 1), \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \rangle \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$= (0, 0, 1) - \left(\frac{1}{\sqrt{3}}\right) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{6}}\right) \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$= (0, 0, 1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(-\frac{2}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

$$= \left(0 - \frac{1}{3} + \frac{2}{6}, 0 - \frac{1}{3} - \frac{1}{6}, 1 - \frac{1}{3} - \frac{1}{6}\right)$$

$$= \left(0, -\frac{2}{6}, \frac{6-2-1}{6}\right)$$

$\omega_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$

$$\|\omega_3\| = \sqrt{0 + \frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$$

$\therefore u_3 = \frac{(0, -\frac{1}{2}, \frac{1}{2})}{\frac{1}{\sqrt{2}}}$

$= \sqrt{2} (0, -\frac{1}{2}, \frac{1}{2}) = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

Thus the orthonormal basis is  $\{u_1, u_2, u_3\} = \left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$