

Eigen Values and Eigen Vectors

Let A be a square matrix over R , of order n then a scalar number ' λ ' is called Eigen Value of A , if there exists a non-zero column vector $V \in R^n$ such that

$$\boxed{AV = \lambda V}$$

Here ' V ' is called Eigen Vector of A corresponding to Eigen value ' λ '
 I is Identity Matrix

$$\begin{aligned} \text{Now } AV = \lambda V &\Rightarrow AV = \lambda IV \\ &\Rightarrow (A - \lambda I)V = 0 \end{aligned}$$

Since $V \neq 0$ (given)

So $A - \lambda I$ is singular Matrix by theorem of homogenous system of eqs

$$\therefore \boxed{|A - \lambda I| = 0} \quad \text{--- (1)}$$

$|A - \lambda I|$ is called characteristic polynomial of matrix A

Every Root of eq (1) ^{ie value of λ} is called Eigen Value of A or Characteristic Value of A

The value of ' V ' corresponding to value of ' λ ' given by $(A - \lambda I)V = 0$ is called Eigen Vector corresponding to λ .

(standard)

Example 15 Find the eigen values and corresponding eigenvectors of the matrix.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\text{Sol } A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \quad \text{--- (1)}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2[2-\lambda-1] + 1[2-(3-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)[6-5\lambda+\lambda^2-2] - 2(1-\lambda) + (-1+\lambda) = 0$$

$$\Rightarrow 12-10\lambda+2\lambda^2-4-6\lambda+5\lambda^2-\lambda^3+2\lambda-2+2\lambda-1+\lambda=0$$

$$\Rightarrow -\lambda^3+7\lambda^2-11\lambda+5=0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$\therefore \lambda = 5, 1, 1 \text{ (Eigenvalues)}$$

Let Eigen Vector 'V' for $\lambda = 5$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$(A - \lambda I)V = 0$$

put $\lambda = 5$
in ① $\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$-3x + 2y + z = 0$$

$$x - 2y + z = 0$$

$$x + 2y - 3z = 0$$

for Solution of Homogeneous Eqs
Reduce in Echelon form

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ -3 & 2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \text{ by } R_{12}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{bmatrix} \text{ by } 3R_1 + R_2 \text{ and } -R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \text{ by } \frac{1}{4}R_2, \frac{1}{4}R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ by } -R_2 + R_3 \text{ (Echelon form)}$$

Rank of $(A^*) = 2 < \text{No. of unknowns} = 3 \therefore$ System has non-trivial sol.

$$x - 2y + z = 0 \quad \text{--- (II)}$$

$$y - z = 0$$

$$\Rightarrow y = z \text{ --- (III) Put in (II)}$$

$$\text{from (II)} \quad x - 2y + y = 0$$

$$x - y = 0 \Rightarrow x = y$$

(Giving Arbitrary Value) let $x = y = z = a$

$$\text{then } V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$$

Eigen Vector corresponding to $\lambda = 5$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [1 \ 1 \ 1]^T$ where $a = 1$ is arbitrary

$$\begin{array}{c|cccc} & 1 & -7 & 11 & -5 \\ 1 & \downarrow & & & \\ & 1 & -6 & 5 & 10 \end{array}$$

$$x^2 - 6x + 5 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot 5}}{2}$$

$$= \frac{6 \pm 4}{2} = 5, 1$$

Homogeneous Eqs $AX = 0$

System of homogeneous
in 'eqs' in 'n' unknowns
has non-trivial sol if
 $\text{Rank } A^* < n$

$$A^* = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now for Eigen Value $\lambda=1$ Eigen vector $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$(A - \lambda I)V = 0$$

Put $\lambda=1$ in ①

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce in Echelon form

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ Echelon form by } -R_1 + R_2, -R_1 + R_3$$

$$\text{Now } \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + 2y + z = 0$$

Let $y=a$ & $z=b$ (give arbitrary values)

$$\text{So } x + 2a + b = 0 \Rightarrow x = -2a - b$$

$$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2a - b \\ a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} -2a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ 0 \\ b \end{bmatrix}$$

$$= a \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Linearly independent
 $\because \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is Echelon form.

Here we have two linearly independent vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}^t$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}^t$ corresponding to eigen vector $\lambda=1$.

Any linear combination of $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}^t$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}^t$ is also an Eigen vector corresponding to $\lambda=1$.

The Set of linear combinations of these eigen vectors form a subspace of R^3 . This subspace is called Eigenspace of A corresponding to $\lambda=1$.

The basis of eigenspace = $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}^t, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}^t \right\}$

Theorem

Non-zero eigen vectors of a matrix A corresponding to distinct eigen values are linearly independent.

Proof Let $v_1, v_2, \dots, v_n$ be non-zero eigen vectors of A corresponding to distinct eigen values $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ respectively.

We prove the theorem by induction.

For $n=1$

then v_1 is linearly independent as $v_1 \neq 0$ so $a=0$.

For $n=k$

suppose the theorem is true i.e. vectors $v_1, v_2, \dots, v_k$ are linearly independent. i.e. $a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0$ — (1)

For $n=k+1$ Consider $a_1 v_1 + a_2 v_2 + \dots + a_k v_k + a_{k+1} v_{k+1} = 0$ — (2)

$$A(a_1 v_1 + a_2 v_2 + \dots + a_k v_k + a_{k+1} v_{k+1}) = A \cdot 0$$

$$\Rightarrow a_1 A v_1 + a_2 A v_2 + \dots + a_k A v_k + a_{k+1} A v_{k+1} = 0$$

$$\Rightarrow a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 + \dots + a_k \lambda_k v_k + a_{k+1} \lambda_{k+1} v_{k+1} = 0 \quad \because Av = \lambda v \text{ by def.} \quad \text{--- (3)}$$

Multiply (2)

$$\text{by } \lambda_{k+1} \Rightarrow a_1 \lambda_{k+1} v_1 + a_2 \lambda_{k+1} v_2 + \dots + a_k \lambda_{k+1} v_k + a_{k+1} \lambda_{k+1} v_{k+1} = 0 \quad \text{--- (4)}$$

Subtracting

$$\text{(4) from (3)} \Rightarrow a_1 (\lambda_1 - \lambda_{k+1}) v_1 + a_2 (\lambda_2 - \lambda_{k+1}) v_2 + \dots + a_k (\lambda_k - \lambda_{k+1}) v_k + 0 = 0$$

From (1) the vectors $v_1, v_2, \dots, v_k$ are linearly independent (assumed)

$$\text{So } a_1 (\lambda_1 - \lambda_{k+1}) = 0$$

$$a_2 (\lambda_2 - \lambda_{k+1}) = 0$$

!

$$a_k (\lambda_k - \lambda_{k+1}) = 0$$

Since $\lambda_1, \lambda_2, \dots, \lambda_k$ are all distinct So $\lambda_i - \lambda_{k+1} \neq 0 \quad i=1, 2, \dots, k$

$$\Rightarrow a_i = 0 \quad i=1, 2, \dots, k$$

using $a_i = 0, i=1, 2, \dots, k$ in (2)

$$\Rightarrow a_{k+1} v_{k+1} = 0$$

$$\Rightarrow a_{k+1} = 0 \quad \because v_{k+1} \text{ is nonzero}$$

Thus vectors $v_1, v_2, \dots, v_{k+1}$ are all linearly independent

Hence by Induction $v_1, v_2, \dots, v_n$ are linearly independent.

Theorem Any two eigenvectors corresponding to two distinct eigen values of an orthogonal matrix are orthogonal.

Proof Let A be an $n \times n$ orthogonal matrix, having eigenvectors v_1, v_2 corresponding to distinct eigen values λ_1, λ_2 .

Then by def $Av_1 = \lambda_1 v_1$ ——— ①

and $Av_2 = \lambda_2 v_2$

$\lambda_1 \neq \lambda_2$ $\because$ distinct eigenvalues (given)

taking transpose

$$(Av_1)^t = (\lambda_1 v_1)^t$$

$$v_1^t A^t = \lambda_1 v_1^t \text{ ——— ②}$$

From ① & ②

$$(v_2^t A^t) Av_1 = (\lambda_2 v_2^t) (\lambda_1 v_1)$$

$$v_2^t (A^t A) v_1 = \lambda_1 \lambda_2 v_2^t v_1$$

$$v_2^t I v_1 = \lambda_1 \lambda_2 v_2^t v_1$$

$$v_2^t v_1 = \lambda_1 \lambda_2 v_2^t v_1$$

$$v_2^t v_1 (1 - \lambda_1 \lambda_2) = 0 \text{ ——— ③}$$

$$v_2^t v_1 (\lambda_2^2 - \lambda_1 \lambda_2) = 0$$

$$v_2^t v_1 \lambda_2 (\lambda_2 - \lambda_1) = 0$$

by theorem
Since λ is an eigen value of an orthogonal matrix then $|\lambda| = 1$ i.e. $\lambda^2 = 1 \therefore \lambda^2 = 1$

As $\lambda_2 \neq \lambda_1$ & $\lambda_2 \neq 0 \therefore |\lambda_2| = 1$

$$\therefore v_2^t v_1 = 0$$

$$\Rightarrow \langle v_2, v_1 \rangle = 0$$

$\because$ (see Example 2) $\perp$ P of v_1, v_2 Col vectors
is $\langle v_1, v_2 \rangle = v_1^t v_2$

Hence v_1, v_2 are orthogonal vectors.

Theorem If λ is an eigen value of an orthogonal matrix, then $|\lambda| = 1$

Proof Let A be a square matrix of order n . Also A is orthogonal matrix and λ be an eigen value of A . Then there exists a non-zero column vector $v \in \mathbb{R}^n$ such that

$$Av = \lambda v \quad \text{--- (1)}$$

taking
transpose

$$(Av)^t = (\lambda v)^t$$

$$v^t A^t = \lambda v^t \quad \text{--- (2)}$$

$\therefore \lambda$ is scalar

From (1) & (2) $(v^t A^t) Av = (\lambda v^t) \lambda v$

$$v^t (A^t A) v = \lambda^2 v^t v$$

$$v^t I v = \lambda^2 v^t v$$

$$v^t v = \lambda^2 v^t v$$

$$v^t v (1 - \lambda^2) = 0$$

$$1 - \lambda^2 = 0$$

$$\therefore v \neq 0 \text{ so } v^t v \neq 0$$

$$1 = \lambda^2$$

$$1 = |\lambda|^2$$

$$1 = |\lambda|$$

Proved

x ----- x

(X both sides of (1) by $A^t v$, then put $Av = \lambda v$ on R.H.S.)

$\therefore A$ is orthogonal
so $A^t A = I$