

Ex 7.3

Q1. For each of the following matrices, find the characteristic polynomial, all eigen values, and a basis of each eigen space.

$$(1) A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

Characteristic Polynomial $|A - \lambda I|$

$$A - \lambda I = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{pmatrix} \quad \text{--- (1)}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore |A - \lambda I| = 0$$

$$\Rightarrow (3-\lambda) \{ (4-\lambda)(3-\lambda) - 2 \} - \{ 6 - 2\lambda - 2 \} + \{ 2 - 4 + \lambda \} = 0$$

$$\Rightarrow (3-\lambda) \{ 12 - 7\lambda + \lambda^2 - 2 \} - 4 + 2\lambda + 2 - 4 + \lambda = 0$$

$$\Rightarrow (3-\lambda)(10 - 7\lambda + \lambda^2) - 6 + 3\lambda = 0$$

$$\Rightarrow 30 - 21\lambda + 3\lambda^2 - 10\lambda + 7\lambda^2 - \lambda^3 - 6 + 3\lambda = 0$$

$$\Rightarrow -\lambda^3 + 10\lambda^2 - 28\lambda + 24 = 0$$

$$|A - \lambda I| = \boxed{\lambda^3 - 10\lambda^2 + 28\lambda - 24} \quad \text{Characteristic Polynomial.}$$

Eigen Values (i.e. Roots) are $\lambda = 2, 2, 6$

$$1 \left| \begin{array}{ccc|c} 1 & -10 & 28 & -24 \\ \downarrow & 1 & -9 & 19 \\ 1 & -9 & 19 & -5 \end{array} \right.$$

$$2 \left| \begin{array}{ccc|c} 1 & -10 & 28 & -24 \\ \downarrow & 2 & -16 & 24 \\ 1 & -8 & 12 & 0 \end{array} \right.$$

$$\lambda^2 - 8\lambda + 12 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64 - 48}}{2}$$

$$\lambda = \frac{8 \pm 4}{2} = 2, 6$$

Let Eigen Vector $\vec{v}$ for $\lambda = 2$ is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(A - \lambda I)\vec{v} = 0$$

Put $\lambda = 2$ in (1)

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now reduce matrix in Echelon form

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{R} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{by } -2R_1 + R_2, -R_1 + R_3$$

$$\text{Now } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x + y + z = 0$$

For solⁿ giving arbitrary values to y & z , for let $y = a, z = b$

$$x + a + b = 0$$

$$x = -a - b$$

$$\begin{aligned} \text{then } v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} -a-b \\ a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} -b \\ 0 \\ b \end{pmatrix} \\ &= +a \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

The Eigen vectors corresponding to $\lambda = 2$ are $[-1 \ 1 \ 0]^t, [-1 \ 0 \ 1]^t$

Basis of Eigenspace corresponding to $\lambda = 2$ is $\{[-1 \ 1 \ 0]^t, [-1 \ 0 \ 1]^t\}$

Now for $\lambda = 6$ Eigen Vector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(A - \lambda I)v = 0$$

$$\text{Put } \lambda = 6 \text{ in } \begin{pmatrix} -3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \text{Now reduce Matrix in Echelon form } \begin{pmatrix} -3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 1 & -3 \end{pmatrix} &\xrightarrow{R} \begin{pmatrix} 1 & 1 & -3 \\ 2 & -2 & 2 \\ -3 & 1 & 1 \end{pmatrix} \xrightarrow{R_{13}} \begin{pmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{pmatrix} \xrightarrow{R} \begin{pmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 4 & -8 \end{pmatrix} \xrightarrow{\substack{-2R_1+R_2 \\ 3R_1+R_3}} \begin{pmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\substack{-R_1 \\ -4R_2+R_3}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$x + y - 3z = 0$$

$$y - 2z = 0 \Rightarrow y = 2z$$

Let $z = a$ be arbitrary value

$$\begin{aligned} x + 2a - 3a &= 0 \Rightarrow x = a \\ y &= 2a \\ z &= a \end{aligned}$$

$$\therefore v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ 2a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$v = [1 \ 2 \ 1]^t$$

Eigen Vector corresponding to $\lambda = 6$ is $[1 \ 2 \ 1]^t$

Basis of Eigenspace corresponding to $\lambda = 6$ is $\{[1 \ 2 \ 1]^t\}$

Q 1(iii) $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{pmatrix} \text{ --- (1)}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(4-\lambda)+1] - 2[(4-\lambda)-1] + 2[1+(2-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)[8-6\lambda+\lambda^2+1] - 2[3-\lambda] + 2[3-\lambda] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-6\lambda+9) = 0$$

$$\Rightarrow \lambda^2-6\lambda+9 - \lambda^3+6\lambda^2-9\lambda = 0$$

$$\Rightarrow -\lambda^3+7\lambda^2-15\lambda+9 = 0$$

$$\Rightarrow \lambda^3-7\lambda^2+15\lambda-9 = 0$$

$$|A - \lambda I| = \boxed{\lambda^3 - 7\lambda^2 + 15\lambda - 9} \text{ Characteristic Polynomial}$$

Eigen Values (ie Roots) are $\lambda = 1, 3, 3$

$$\begin{array}{c|ccc} & 1 & -7 & 15 & -9 \\ & \downarrow & & & \\ & 1 & -6 & 9 & 0 \end{array}$$

For $\lambda = 1$

Let Eigen Vector $V = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(A - \lambda I)V = 0$$

Put $\lambda = 1$ in (1)

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now reduce matrix in Echelon form

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{pmatrix} \xrightarrow{R} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \xrightarrow{\text{by } R_{12}} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \xrightarrow{\text{by } R_1 + R_3}$$

$$\xrightarrow{R} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \xrightarrow{\text{by } \frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{Echelon form}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{by } -2R_2 + R_1}$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36-36}}{2}$$

$$\lambda = \frac{6 \pm 0}{2} = 3, 3$$

Now
$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x + y - z = 0$$

$$y + z = 0 \Rightarrow y = -z$$

Let $z = a$ be arbitrary value

$$x + (-a) - a = 0 \Rightarrow x = 2a$$

$$y = -a$$

$$z = a$$

$$\therefore v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2a \\ -a \\ a \end{pmatrix} = a \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Eigen Vector $v = [2 \ -1 \ 1]^t$ corresponding to $\lambda = 1$

Basis of Eigen Space corresponding to Eigen Value $\lambda = 1$ is $\{[2 \ -1 \ 1]^t\}$

Now for $\lambda = 3$ Eigen Vector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(A - \lambda I)v = 0$$

Put $\lambda = 3$ in (1)

$$\begin{pmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now

Reduce Matrix in Echelon form

$$\begin{pmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \xrightarrow{R} \begin{pmatrix} 1 & -1 & -1 \\ -2 & 2 & 2 \\ -1 & 1 & 1 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{2R_1 + R_2, R_1 + R_3}$$

Now

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x - y - z = 0$$

Let $y = a$, $z = b$ be arbitrary values.

$$\therefore \Rightarrow x = a + b$$

Hence
$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+b \\ a \\ b \end{pmatrix} = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} b \\ 0 \\ b \end{pmatrix}$$

$$= a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Therefore the Eigen Vectors corresponding to Eigen Value $\lambda = 3$ are $[1 \ 1 \ 0]^t$ & $[1 \ 0 \ 1]^t$

Basis of Eigen Space corresponding to $\lambda = 3$ is $\{[1 \ 1 \ 0]^t, [1 \ 0 \ 1]^t\}$

(2) (iii) $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix} \quad \text{--- ①}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) \left[(4-\lambda)(-5-\lambda) + 18 \right] + 3 \left[3(4-\lambda) - 18 \right] + 3 \left[-18 + 6(5+\lambda) \right] = 0$$

$$\Rightarrow (1-\lambda) \left[-20 + \lambda + \lambda^2 + 18 \right] + 3 \left[12 - 3\lambda - 18 \right] + 3 \left[12 + 6\lambda \right] = 0$$

$$\Rightarrow (1-\lambda) \left[\lambda^2 + \lambda - 2 \right] + 3 \left[-6 - 3\lambda \right] + (36 + 18\lambda) = 0$$

$$\Rightarrow \cancel{\lambda^2} + \lambda - 2 - \lambda^3 - \cancel{\lambda^2} + 2\lambda - 18 - 9\lambda + 36 + 18\lambda = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda + 16 = 0$$

$$|A - \lambda I| = \boxed{\lambda^3 - 12\lambda - 16} \quad \text{Characteristic Polynomial}$$

Eigen Values (i.e) Roots are $\lambda = 4, -2, -2$

$$4 \left| \begin{array}{ccc|c} 1 & 0 & -12 & -16 \\ \hline \downarrow & 4 & 16 & 16 \\ 1 & 4 & 4 & 0 \end{array} \right|$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-16}}{2}$$

$$= \frac{-4 \pm 0}{2} = -2, -2$$

For $\lambda = 4$ Let Eigenvector $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$(A - \lambda I)V = 0$$

Put $\lambda = 4$ in ①

$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce
Matrix in
Echelon
Form

$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \xrightarrow{R} \begin{pmatrix} 1 & 1 & -1 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \xrightarrow{\text{by } -\frac{1}{3}R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{pmatrix} \xrightarrow{\text{by } -3R_1 + R_2, -6R_1 + R_3}$$

$$R \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -12 & 6 \end{pmatrix} \xrightarrow{-\frac{1}{12}R_3} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Now } \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x + y - z = 0$$

$$y - \frac{z}{2} = 0 \Rightarrow y = \frac{z}{2}$$

Let $z = a$ be arbitrary value

$$\therefore y = \frac{a}{2}$$

$$\text{and } x + \frac{a}{2} - a = 0 \Rightarrow x = a - \frac{a}{2} \\ x = \frac{a}{2}$$

$$\therefore v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a/2 \\ a/2 \\ a \end{pmatrix} = a \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

Eigen Vector $v = \left[\frac{1}{2} \frac{1}{2} 1 \right]^t$ corresponding to $\lambda = 4$

Basis of Eigen Space corresponding to Eigen Value $\lambda = 4$ is $\left\{ \left[\frac{1}{2} \frac{1}{2} 1 \right]^t \right\}$

Now for $\lambda = -2$ Eigen Vector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(A - \lambda I)v = 0$$

$$\text{Put } \lambda = -2 \quad \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Now reduce Matrix in Echelon Form } \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \xrightarrow{R} \begin{pmatrix} 1 & -1 & 1 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\substack{-3R_1+R_2 \\ -6R_1+R_3}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ Echelon form}$$

$$\text{Now } \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x - y + z = 0$$

Let $y = a$, $z = b$ be arbitrary values

$$\therefore x - a + b = 0 \Rightarrow x = a - b$$

$$\text{Hence } v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a-b \\ a \\ b \end{pmatrix} = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} -b \\ 0 \\ b \end{pmatrix} \\ = a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Therefore the Eigen Vectors corresponding to Eigen Value $\lambda = -2$ are $[1 \ 1 \ 0]^t$ and $[-1 \ 0 \ 1]^t$

Basis of Eigen Space corresponding to $\lambda = -2$ is $\left\{ [1 \ 1 \ 0]^t, [-1 \ 0 \ 1]^t \right\}$

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Q1 (iv) $A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{bmatrix} \quad \text{--- ①}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-3-\lambda)(5-\lambda)(-2-\lambda) + 6 - 1[-7(-2-\lambda) - 6] - [-42 + 6(5-\lambda)] = 0$$

$$\Rightarrow (-3-\lambda)(-10-5\lambda+2\lambda+\lambda^2+6) - 1[+14+7\lambda-6] - [-42+30-6\lambda] = 0$$

$$\Rightarrow (-3-\lambda)(-4-3\lambda+\lambda^2) - 8-7\lambda+12+6\lambda = 0$$

$$\Rightarrow 12 + 9\lambda - 3\lambda^2 + 4\lambda + 3\lambda^2 - \lambda^3 + 4 - \lambda = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda + 16 = 0$$

$$|A - \lambda I| = \boxed{\lambda^3 - 12\lambda - 16} \text{ Characteristic Polynomial}$$

Eigen Values are $\lambda = 4, -2, -2$,

For $\lambda = 4$ Eigen Vectors $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$(A - \lambda I)V = 0$$

Put $\lambda = 4$

$$\text{u ① } \begin{bmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce
Matrix in
Echelon form

$$\begin{bmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} -7 & 1 & -1 \\ 0 & 0 & 0 \\ -6 & 6 & -6 \end{bmatrix} \xrightarrow{R_3 \div (-6)} \begin{bmatrix} -7 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 1 \\ -7 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 + 7R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \div (-6)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ Echelon form}$$

Now $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$x - y + z = 0$$

$$y - z = 0 \Rightarrow y = z$$

Let $z = a$ be the arbitrary value of
 $y = a$

$$x - a + a = 0 \Rightarrow x = 0$$

Hence $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ a \end{pmatrix} = a \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ Eigen vector corresponding to $\lambda = 4$ is $[1 \ 1 \ 1]^t$

Basis of Eigen Space corresponding to $\lambda = 4$ is $\{[1 \ 1 \ 1]^t\}$

For $\lambda = -2$ Eigen vector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(A - \lambda I)v = 0$$

Put $\lambda = -2$
 $\Rightarrow$ $\begin{pmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Now reduce
 Matrix -
 Echelon Form

$$\begin{pmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix} \xrightarrow{R_1} \begin{pmatrix} 1 & -1 & 1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix} \xrightarrow{R_2, R_3} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{7R_1 + R_2, 6R_1 + R_3} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \div 6} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-6R_2 + R_3} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Now $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow x - y + z = 0$$

$$z = 0$$

$$\Rightarrow x - y + 0 = 0 \Rightarrow x = y$$

Let $y = a \Rightarrow x = a$

Hence $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Eigen vector corresponding to $\lambda = -2$ is $[1 \ 1 \ 0]^t$

Basis of Eigen Space corresponding to $\lambda = -2$ is $\{[1 \ 1 \ 0]^t\}$

x _____

Q2 Show that eigen values of a diagonal matrix are its diagonal elements and the eigen vectors are the standard basis vectors.

Sol Let $A = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$

Diagonal Matrix

In diagonal matrix diagonal elements are non zero.

$$A - \lambda I = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} a_{11}-\lambda & 0 & 0 & \dots & 0 \\ 0 & a_{22}-\lambda & 0 & \dots & 0 \\ 0 & 0 & a_{33}-\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn}-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & 0 & 0 & \dots & 0 \\ 0 & a_{22}-\lambda & 0 & \dots & 0 \\ 0 & 0 & a_{33}-\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) \dots (a_{nn}-\lambda) = 0$$

$$\Rightarrow (a_{11}-\lambda) = 0, (a_{22}-\lambda) = 0, \dots, (a_{nn}-\lambda) = 0$$

$$\Rightarrow \lambda = a_{11}, a_{22}, a_{33}, \dots, a_{nn} \quad \text{ie Eigen Values are diagonal elements}$$

Since this is determinant of triangular matrix, so it is expanded by product of diagonal elements

Now For $\lambda = a_{11}$ Eigen Vector $v_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

$$(A - \lambda I) v_1 = 0$$

$$\Rightarrow \begin{pmatrix} a_{11}-a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22}-a_{11} & 0 & \dots & 0 \\ 0 & 0 & a_{33}-a_{11} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn}-a_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Rightarrow a_{11}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \dots + 0 \cdot x_n = 0$$

$$0 \cdot x_1 + (a_{22}-a_{11})x_2 + 0 \cdot x_3 + \dots + 0 \cdot x_n = 0$$

$$0 \cdot x_1 + 0 \cdot x_2 + (a_{33}-a_{11})x_3 + \dots + 0 \cdot x_n = 0$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \dots + (a_{nn}-a_{11})x_n = 0$$

$$\Rightarrow 0 \cdot x_1 = 0$$

$$\Rightarrow (a_{22}-a_{11})x_2 = 0$$

$$\Rightarrow (a_{33}-a_{11})x_3 = 0$$

$$\Rightarrow (a_{nn}-a_{11})x_n = 0$$

$$\Rightarrow x_1 \text{ is non-zero,}$$

$$\Rightarrow x_2 = 0 \quad \text{as } (a_{22}-a_{11}) \neq 0 \text{ as diagonal elements are non-zero}$$

$$\Rightarrow x_3 = 0 \quad \text{as } (a_{33}-a_{11}) \neq 0 \text{ as diagonal elements are non-zero}$$

$$\Rightarrow x_n = 0 \quad \text{as } (a_{nn}-a_{11}) \neq 0 \text{ as diagonal elements are non-zero}$$

Let $x_1 = a$ giving arbitrary value

$$\text{Hence } v_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow v_1 = [1 \ 0 \ 0 \ \dots \ 0]^t$$

Similarly for $\lambda = a_{22}, a_{33}, \dots, a_{nn}$

$$\text{we have } v_2 = [0 \ 1 \ 0 \ \dots \ 0]^t, v_3 = [0 \ 0 \ 1 \ \dots \ 0]^t, \dots, v_n = [0 \ 0 \ \dots \ 1]^t$$

Hence the Eigen Vectors $v_1, v_2, v_3, \dots, v_n$ are Standard basis for $\mathbb{R}^n$

Q3: Show that A and A^t have the same eigen values. Give an example where A and A^t have different eigen vectors (A is a square Matrix)

Sol Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$
Square Matrix

Eigen Values of A are given by $|A - \lambda I| = 0$

$$\therefore A - \lambda I = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & 0 & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} - \lambda \end{vmatrix} = 0 \quad \text{--- (I)}$$

Now $A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \dots & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nn} \end{pmatrix}$

Eigen values of A^t are given by $|A^t - \lambda I| = 0$

$$|A^t - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{21} & a_{31} & \dots & a_{n1} \\ a_{12} & a_{22} - \lambda & a_{32} & \dots & a_{n2} \\ a_{13} & a_{23} & a_{33} - \lambda & \dots & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nn} - \lambda \end{vmatrix} = 0 \quad \text{--- (II)}$$

We know that the value of determinant is unchanged if rows and columns are interchanged (i.e. transpose)

$$\therefore \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & 0 & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} - \lambda \end{vmatrix} = \begin{vmatrix} a_{11} - \lambda & a_{21} & a_{31} & \dots & a_{n1} \\ a_{12} & a_{22} - \lambda & a_{32} & \dots & a_{n2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nn} - \lambda \end{vmatrix}$$

$$|A - \lambda I| = |A^t - \lambda I|$$

Hence Eigen Values of A & A^t are same.

Example of different Eigen Vectors of A & A^t

Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$

For Eigen Value $|A - \lambda I| = 0$ For Eigen Vector $(A - \lambda I)v = 0$

$$A - \lambda I = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 4 \\ 1 & 5-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 4 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$$

$$\Rightarrow 10 - 7\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$$

$$\Rightarrow \lambda = \frac{7 \pm \sqrt{49 - 24}}{2} = \frac{7 \pm 5}{2}$$

$$\Rightarrow \lambda = 6, 1 \text{ Eigen Values}$$

For $\lambda = 1$ Eigen Vector of A is given by $(A - \lambda I)v = 0$

Put $\lambda = 1$ in ① $\Rightarrow \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x + 4y = 0 \Rightarrow x = -4y \Rightarrow x = -4a \text{ where } a \text{ is arbitrary}$$

$$x + 4y = 0 \Rightarrow -4a + 4y = 0 \Rightarrow y = \frac{4a}{4} = a$$

$$\therefore v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4a \\ a \end{bmatrix} = a \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$v = \begin{bmatrix} -4 \\ 1 \end{bmatrix}^t \quad \text{--- ③}$$

For $\lambda = 1$ the Eigen Vector of A^t is given by $(A^t - \lambda I)v = 0$

$$A^t - \lambda I = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{bmatrix} \quad \text{--- ②}$$

$$|A^t - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - 4 \Rightarrow \lambda = 6, 1 \text{ Eigen Values}$$

$$(A^t - \lambda I)v = 0$$

Put $\lambda = 1$ in ②

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x + y = 0 \Rightarrow x = -y = -a$$

 $\Rightarrow x = -a$
arbitrary value.

$$\therefore v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -a \\ a \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}^t \quad \text{--- ④}$$

from ③
& ④Eigen Vector for A & A^t are different. $\longleftarrow x$

- ④ Suppose v is an eigenvector of a square matrix A corresponding to the eigenvalue λ . Show that for $n > 0$, v is also an eigenvector of A^n corresponding to λ^n .

(given) $Av = \lambda v$ — ①

To Prove $A^n v = \lambda^n v$

We prove by induction

For $n=1$ $Av = \lambda v$ true by ① C-1 is satisfied

For $n=k$ $A^k v = \lambda^k v$ true supposed. — ②

Consider $A^{k+1} v = A(A^k v)$

$= A(\lambda^k v)$

$= \lambda^k (Av)$

$= \lambda^k (\lambda v)$

$A^{k+1} v = \lambda^{k+1} v$

using ②

$\therefore \lambda^k$ is scalar

using ①

C-2 is satisfied

Hence the result is true for all the integral values of n .

- ⑤ If λ is an eigen value of a non-singular matrix A , then λ^{-1} is an eigen value of A^{-1} .

id We know

$Av = \lambda v$

$A^{-1}(Av) = A^{-1}(\lambda v)$

$(A^{-1}A)v = \lambda(A^{-1}v)$

$Iv = \lambda(A^{-1}v)$

$v = \lambda(A^{-1}v)$

$\lambda^{-1}v = A^{-1}v$

Hence λ^{-1} is the eigen value of A^{-1} .

(given)

$\therefore \lambda$ is an eigen value of A

A is non singular matrix implies A^{-1} exists.

Let v is eigenvector corresponding to eigen value λ .

$\therefore \lambda$ is scalar no.

⑤ If A & B are square matrices show that AB & BA have same eigen values.

Sol $|A - \lambda I| = 0$ gives the Eigen values λ of A .

$\therefore |AB - \lambda I| = 0$ gives the Eigen values of AB

$$\Rightarrow |B^{-1}B(AB) - B^{-1}B(\lambda I)| = 0 \quad \times \text{ by } B^{-1}$$

$$\Rightarrow |B^{-1}(BA)B - \lambda(B^{-1}B)I| = 0$$

$$\Rightarrow |B^{-1}(BA - \lambda I)B| = 0$$

$$\Rightarrow |B^{-1}| |BA - \lambda I| |B| = 0 \quad (\text{By Product Th})$$

$$\Rightarrow \frac{1}{|B|} |BA - \lambda I| |B| = 0 \quad \because \det A^{-1} = \frac{1}{\det A}$$

$$\Rightarrow |BA - \lambda I| = 0 \quad \text{gives Eigen values } \lambda \text{ of } BA.$$

⑥ 2nd Method

If λ be an eigen value of A ,
corresponding to eigen vector V

$$\text{then } AV = \lambda V$$

Let λ be an eigen value of AB . then $(AB)V = \lambda V$

$$B(AB)V = B(\lambda V) \quad \times \text{ by } B$$

$$(BA)(BV) = \lambda(BV)$$

which shows that BV is an eigen vector of BA with eigen value λ . Thus eigen values of BA & AB are same.

⑦ If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigen values of a square matrix of order 'n' then $K\lambda_1, K\lambda_2, \dots, K\lambda_n$, where K is a scalar, are eigen values of KA .

Sol If λ be an eigen value of A
corresponding to eigen vector V

$$\text{then } AV = \lambda V$$

If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigen values of A
of order 'n' corresponding to eigen vector V

$$\text{then } AV = (\lambda_1, \lambda_2, \dots, \lambda_n)V$$

$$KA \cdot V = K(\lambda_1, \lambda_2, \dots, \lambda_n)V$$

$$(KA)V = (K\lambda_1, K\lambda_2, \dots, K\lambda_n)V$$

which shows $K\lambda_1, K\lambda_2, \dots, K\lambda_n$ are eigen values of KA .

⑧ Suppose v is an eigen vector of $n \times n$ matrices A & B . Show that v is also an eigen vector of $aA + bB$, where a & b are any scalar.

Sol v is the eigenvector of A & B corresponding to eigen value λ_1 & λ_2

$$\text{then } Av = \lambda_1 v \quad \text{and} \quad Bv = \lambda_2 v$$

$$\Rightarrow aAv = a\lambda_1 v \quad \text{--- (i)} \quad bBv = b\lambda_2 v \quad \text{--- (ii)}$$

Add (i) & (ii)

$$aAv + bBv = a\lambda_1 v + b\lambda_2 v$$

$$(aA + bB)v = (a\lambda_1 + b\lambda_2)v$$

$\Rightarrow v$ is the eigen vector of matrix $aA + bB$

⑨ Let λ be an eigen value of a square matrix A . Let V_λ denote the set of all eigen vectors of A corresponding to eigen value λ . Show that V_λ is a subspace of V (V_λ is the eigen space of A corresponding to λ)

Sol To prove V_λ is subspace of V .

Let v_1, v_2 belong to V_λ

a_1, a_2 be scalars then

$$Av_1 = \lambda v_1 \quad \text{--- (i)}$$

$$Av_2 = \lambda v_2 \quad \text{--- (ii)}$$

$$\begin{aligned} \text{and } A(a_1 v_1 + a_2 v_2) &= a_1 Av_1 + a_2 Av_2 \\ &= a_1 \lambda v_1 + a_2 \lambda v_2 \\ &= \lambda (a_1 v_1 + a_2 v_2) \end{aligned}$$

So $a_1 v_1 + a_2 v_2$ is an eigen vector of A corresponding to eigen value ' λ '.

Thus $a_1 v_1 + a_2 v_2$ belongs to V_λ .

Hence V_λ is a subspace of V . (Since if $w, w_2 \in W$ and $a, b \in F$ imply $aw_1 + bw_2 \in W$ then W is a subspace of V , where $W \subseteq V$)

Q10 For the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, find all eigenvalues, eigenvectors and a basis for each eigen space.

(i) $T(x, y) = (3x + 3y, x + 5y)$

Standard Basis for $\mathbb{R}^2 = \{e_1 = (1, 0), e_2 = (0, 1)\}$

$T(e_1) = T(1, 0) = (3 \cdot 1 + 3 \cdot 0, 1 + 5 \cdot 0)$

$= (3, 1) = 3e_1 + e_2$

$T(e_2) = T(0, 1) = (3 \cdot 0 + 3 \cdot 1, 0 + 5 \cdot 1)$

$= (3, 5) = 3e_1 + 5e_2$

Matrix of linear transformation, i.e. T is

$\begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix} = A \text{ (say)}$

$A - \lambda I = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & 3 \\ 1 & 5-\lambda \end{pmatrix}$

— ①

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda & 3 \\ 1 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(5-\lambda) - 3 = 0$

$\Rightarrow 12 - 8\lambda + \lambda^2 = 0$

$\Rightarrow \lambda = 6, 2 \text{ Eigen Values}$

For $\lambda = 6$ let Eigen Vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

put $\lambda = 6$ in ①

$\Rightarrow -3x + 3y = 0 \Rightarrow x = y$

$x - y = 0$

$\therefore v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = 2$

$(A - \lambda I)v = 0$

$\Rightarrow \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{matrix} x + 3y = 0 \\ x + 3y = 0 \end{matrix} \Rightarrow x = -3y \text{ Let } y = a \text{ any arbitrary value}$

$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3a \\ a \end{bmatrix} = a \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

We first find matrix of given linear transformation w.r.t standard basis, $e_1 = (1, 0)$
 $e_2 = (0, 1)$

$\lambda^2 - 8\lambda + 12 = 0$

$\lambda = \frac{8 \pm \sqrt{64 - 4 \cdot 1 \cdot 12}}{2}$

$= \frac{8 \pm \sqrt{16}}{2}$

$= 6, 2$

Let $x = a$ any arbitrary value.

$\therefore$ Eigen Vector for $\lambda = 6$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}^t$

The set $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}^t\}$ generates the subspace of $\mathbb{R}^2$ known as Eigen Space of A .
Basis of Eigen Space $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}^t\}$

$\therefore$ Eigen Vector for $\lambda = 2$ is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}^t$

Any linear combination of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}^t$ is also Eigen Vector for $\lambda = 2$. The set $\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}^t\}$ i.e. set of linear combinations is a subspace of $\mathbb{R}^2$ called Eigen Space of A having basis $\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}^t\}$

(ii) For the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ find all eigenvalues, eigenvectors and Basis for eigenspace.

$$T(x, y) = (y, x)$$

$$T(e_1) = T(1, 0) = (0, 1) = 0 \cdot e_1 + e_2$$

$$T(e_2) = T(0, 1) = (1, 0) = 1 \cdot e_1 + 0 \cdot e_2$$

$e_1 = (1, 0)$
 $e_2 = (0, 1)$ standard Basis for $\mathbb{R}^2$

Matrix of T is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A$
 i.e. Matrix of linear Transformation

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \rightarrow 0$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$\lambda^2 = 1 \Rightarrow \lambda = 1, -1 \text{ Eigenvalues.}$$

For $\lambda = 1$

$$(A - \lambda I)V = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 1$ in 0

$\Rightarrow$

$$-x + y = 0$$

$$x - y = 0$$

$$\Rightarrow y = x$$

let $y = a$, arbitrary value.
 $\therefore x = a$

$$\therefore V = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvector corresponding to $\lambda = 1$
 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}^t$

Any linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}^t$ is also the eigenvector for $\lambda = 1$.
 The set $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}^t \right\}$ is set of linear combinations is a subspace of $\mathbb{R}^2$ called Eigenspace having basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}^t \right\}$.

For $\lambda = -1$

$$(A - \lambda I)V = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Put $\lambda = -1$ in 0

$$\Rightarrow x + y = 0$$

$$x + y = 0$$

$$\Rightarrow x = -y$$

let $y = a$ arbitrary value.
 $\therefore x = -a$

$$\therefore V = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -a \\ a \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigenvector corresponding to $\lambda = -1$
 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}^t$

Any linear combination of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}^t$ is also eigenvector for $\lambda = -1$.
 The set $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}^t \right\}$ is set of linear combinations is a subspace of $\mathbb{R}^2$ called Eigenspace having basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}^t \right\}$.

Q10 (iii) For the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ find all eigen values and a basis for each eigen space.

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$$T(x, y) = (y, -x)$$

$$T(e_1) = T(1, 0) = (0, -1) = 0e_1 + e_2$$

$$T(e_2) = T(0, 1) = (1, 0) = 1e_1 + 0e_2$$

Matrix of linear transformation is $T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A$

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \quad \text{--- ①}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

$$\lambda = \pm i$$

Eigen Values are not real i.e. $\pm i$

Q11 (i) For each of the following operators $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, find all eigen values and a basis for each eigen space.

$$(i) T(x, y, z) = (x + y + z, 2y + z, 2y + 3z)$$

$$T(1, 0, 0) = (1 + 0 + 0, 2(0) + 0, 2(0) + 3(0)) = (1, 0, 0) = 1e_1 + 0e_2 + 0e_3$$

$$T(0, 1, 0) = (0 + 1 + 0, 2(1) + 0, 2(1) + 3(0)) = (1, 2, 2) = 1e_1 + 2e_2 + 2e_3$$

$$T(0, 0, 1) = (0 + 0 + 1, 2(0) + 1, 2(0) + 3(1)) = (1, 1, 3) = 1e_1 + 1e_2 + 3e_3$$

Standard Basis for $\mathbb{R}^3$

$$e_1 = (1, 0, 0)$$

$$e_2 = (0, 1, 0)$$

$$e_3 = (0, 0, 1)$$

$$\Rightarrow \text{Matrix of } T \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} = A$$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 2 & 3-\lambda \end{bmatrix} \quad \text{--- ①}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda) \{ (2-\lambda)(3-\lambda) - 2 \} - 1(0-0) + 1(0-0) = 0$$

$$\Rightarrow (1-\lambda)(4-5\lambda+\lambda^2) = 0$$

$$\Rightarrow (1-\lambda)(\lambda-1)(\lambda-4) = 0$$

Eigen Values are $\lambda = 1, 1, 4$

Let Eigen Vector 'V' for $\lambda = 1$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$(A - \lambda I)V = 0 \Rightarrow$$

Put $\lambda = 1$
in (1)

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + y + z = 0 \Rightarrow y = -z$$

$$0x + y + z = 0$$

$$0x + 2y + 2z = 0$$

$$\Rightarrow y = -z \quad \text{let } z = b \neq x = a$$

(arbitrary values)

$$\therefore V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ -b \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -b \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Eigen Vectors corresponding to $\lambda = 1$ are $[1 \ 0 \ 0]^t$, $[0 \ -1 \ 1]^t$

These eigen vectors generates the subspace of R^3 called Eigen Space have basis $\{[1 \ 0 \ 0]^t, [0 \ -1 \ 1]^t\}$

Now Eigen Vector 'V' for $\lambda = 4$ $(A - \lambda I)V = 0$

Put $\lambda = 4$
in (1)

$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x + y + z = 0 \Rightarrow 3x = y + z \Rightarrow x = \frac{y+z}{3}$$

$$-2y + z = 0 \Rightarrow z = 2y$$

$$2y - z = 0 \Rightarrow z = 2y \quad \text{Let}$$

$$\text{Let } x = a \therefore V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ 2a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Eigen Vector corresponding to $\lambda = 4$ is $[1 \ 1 \ 2]^t$

Eigen Vector generates a subspace of R^3 called

Eigen Space of R^3 corresponding to $\lambda = 4$

having basis $\{[1 \ 1 \ 2]^t\}$

2nd Method to find Eigen Vector

$$(A - \lambda I)V = 0$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$
 $R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$y + z = 0$$

$$\text{let } z = b$$

$$\text{let } x = a$$

$$\therefore V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ -b \\ b \end{bmatrix}$$

2nd Method

$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-3x + y + z = 0$$

$$-2y + z = 0$$

$$\Rightarrow z = 2y$$

$$\Rightarrow x = y$$

$$\text{let } x = a$$

$$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ 2a \end{bmatrix} =$$

$$= a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$