

PERMUTATIONS

Permutation

Let X be non-empty set. Let $f: X \rightarrow X$ be a bijective function, then f is called a permutation on X .

If f is a permutation on $X = \{1, 2, 3, \dots, n\}$ then f is written as

$$f = \begin{pmatrix} 1 & 2 & \dots & n \\ (1)f & (2)f & \dots & (n)f \end{pmatrix}$$

Example

Let $X = \{1, 2, 3\}$

Let $f: X \rightarrow X$ be defined by

$$(1)f = 2$$

$$(2)f = 3$$

$$(3)f = 1$$

Clearly f is a bijective function.

Thus f is a permutation on $X = \{1, 2, 3\}$ and is written as

$$f = \begin{pmatrix} 1 & 2 & 3 \\ (1)f & (2)f & (3)f \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Note

- (i) The total number of permutations defined on a set $X = \{1, 2, 3, \dots, n\}$ with n elements is $n!$
- (ii) The set of all permutations defined on a set X with n elements is denoted by S_n
 $\therefore$ Number of elements of $S_n = n!$

Let $X = \{1, 2, 3\}$

Then number of permutations defined on a set X is $3!$ i.e. 6 and are given by

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Thus $S_3 = \{I, f_1, f_2, f_3, f_4, f_5\}$

- (iii) $I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ is called identity permutation.

Composition of Permutations

or Multiplication

$$\text{Let } \alpha = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ (1)\alpha & (2)\alpha & (3)\alpha & \dots & (n)\alpha \end{pmatrix}$$

$$\text{And } \beta = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ (1)\beta & (2)\beta & (3)\beta & \dots & (n)\beta \end{pmatrix}$$

The composition of α and β is denoted by $\alpha \circ \beta$ and is defined by

$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ (1)(\alpha \circ \beta) & (2)(\alpha \circ \beta) & (3)(\alpha \circ \beta) & \dots & (n)(\alpha \circ \beta) \end{pmatrix}$$

$$\text{Where } (x)(\alpha \circ \beta) = ((x)\alpha)\beta$$

$$\text{Let } \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\begin{aligned} \text{Then } \alpha \circ \beta &= \begin{pmatrix} 1 & 2 & 3 \\ (1)(\alpha \circ \beta) & (2)(\alpha \circ \beta) & (3)(\alpha \circ \beta) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ ((1)\alpha)\beta & ((2)\alpha)\beta & ((3)\alpha)\beta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ (2)\beta & (3)\beta & (1)\beta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \end{aligned}$$

Practical Application

$$\text{Let } \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Write α and β in such a way that 2nd row of α and 1st row of β are same.

$$\text{i.e. } \alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

Now ignore 2nd row of α and 1st row of β .

$$\text{i.e. } \alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Note

- (i) Some times we write composition of α and β by $\alpha\beta$ instead of $\alpha \circ \beta$.
- (ii) Multiplication or product or composition of α and β are same.

Theorem

The set S_n of all permutations defined on a set X with n elements is a group under the operation of composition of permutations.

Proof:

- (I) Let $f, g \in S_n$. We shall prove that $f \circ g \in S_n$

Since f, g are permutations on X .

$\therefore f: X \rightarrow X$ and $g: X \rightarrow X$ are bijective functions then $f \circ g: X \rightarrow X$ is also a bijective function.

$\therefore f \circ g$ is a permutation on X .

Hence $f \circ g \in S_n$

- (II) Let $f, g, h \in S_n$. We shall prove that $(f \circ g) \circ h = f \circ (g \circ h)$

$$\begin{aligned} (x)[(f \circ g) \circ h] &= ((x)(f \circ g))h & (x)[f \circ (g \circ h)] &= ((x)f)(g \circ h) \\ &= (((x)f)g)h & &= (((x)f)g)h \end{aligned}$$

Clearly $(f \circ g) \circ h = f \circ (g \circ h)$

(III) Let $I: X \rightarrow X$ be defined by $(x)I = x$

Then $I: X \rightarrow X$ is a bijective function thus I is a permutation on X .

$$\therefore I \in S_n$$

We shall prove that I is an identity element of S_n .

Let $f \in S_n$. We have to show that $f \circ I = I \circ f = f$

$$\begin{aligned} (x)(f \circ I) &= ((x)f)I & (x)(I \circ f) &= ((x)I)f \\ &= (x)f & &= (x)f \end{aligned}$$

$$\text{Clearly } f \circ I = I \circ f = f$$

(IV) Let $f \in S_n$ then $f: X \rightarrow X$ is a bijective function.

$\therefore f^{-1}: X \rightarrow X$ defined as $(x)f = y \Leftrightarrow (y)f^{-1} = x$ is bijective and

hence a permutation on X . $\therefore f^{-1} \in S_n$

Now we shall prove that f^{-1} is the inverse of f and for this we have

to prove that $f \circ f^{-1} = f^{-1} \circ f = I$

$$\begin{aligned} (x)(f \circ f^{-1}) &= ((x)f)f^{-1} & (y)(f^{-1} \circ f) &= ((y)f^{-1})f \\ &= (y)f^{-1} & &= (x)f \\ &= x & &= y \\ &= (x)I & &= (y)I \end{aligned}$$

$$\text{Clearly } f \circ f^{-1} = f^{-1} \circ f = I$$

Hence $(S_n, \circ)$ is a group.

Symmetric Group

The group S_n of all permutation defined on a set X with n elements is called symmetric group of degree n .

Example

Show that $(S_3, \circ)$ is a non-abelian group.

Solution

Here S_3 is the set of all permutation defined on a set X with three elements and is group under the composition of permutations.

If $X = \{1, 2, 3\}$, then all the permutations defined on the set X are

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\begin{aligned} \text{Now } f_1 \circ f_3 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \end{aligned}$$

$$\text{And } f_3 \circ f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Clearly $f_1 \circ f_3 \neq f_3 \circ f_1$

$\Rightarrow (S_3, \circ)$ is a non-abelian group.

Example

$$\text{Let } f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

Show that $f \circ g \neq g \circ f$

Solution

$$\text{Here } f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \\ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 4 & 3 & 2 \end{pmatrix} \\ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

$$\text{And } g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix} \\ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

Clearly $f \circ g \neq g \circ f$

Example

$$\text{Let } \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 6 & 4 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix},$$

Show that $\alpha \circ \beta \neq \beta \circ \alpha$

Solution

$$\text{Here } \alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 6 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix} \\ = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 6 & 4 & 1 \end{pmatrix} \begin{pmatrix} 5 & 3 & 2 & 6 & 4 & 1 \\ 6 & 1 & 4 & 5 & 2 & 3 \end{pmatrix} \\ = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 4 & 5 & 2 & 3 \end{pmatrix}$$

$$\text{And } \beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 6 & 4 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix} \begin{pmatrix} 3 & 4 & 1 & 2 & 6 & 5 \\ 2 & 6 & 5 & 3 & 1 & 4 \end{pmatrix} \\ = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 5 & 3 & 1 & 4 \end{pmatrix}$$

Clearly $\alpha \circ \beta \neq \beta \circ \alpha$

CYCLES

Cycle

Let $X = \{a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_n\}$

If α is a permutation on X , such that

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_k & a_{k+1} & \dots & a_n \\ a_2 & a_3 & \dots & a_1 & a_{k+1} & \dots & a_n \end{pmatrix} \quad \checkmark$$

Then α is called a cyclic permutation or simply a cycle of length k .

Thus α will be called a cycle of length k if

$$(a_1) \alpha = a_2$$

$$(a_2) \alpha = a_3$$

.....

.....

$$(a_k) \alpha = a_1$$

$$(x) \alpha = x \quad \forall x \in X - \{a_1, a_2, \dots, a_k\}$$

The cycle α of length k can also be written as

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ a_2 & a_3 & \dots & a_1 \end{pmatrix}$$

In short α can be written as $\alpha = (a_1 a_2 \dots a_k)$

Here $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$ is a cycle of length 6 which can also

be written as $\alpha = (1 2 3 4 5 6)$

And $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 3 & 6 \end{pmatrix}$ is a cycle of length 3 which can also

be written as $\beta = (3 4 5)$

Example

Show by means of an example that the product of two cyclic permutations need not be a cyclic permutation.

Solution

Let $\alpha = (1 2 5)$ and $\beta = (2 1 4 5 6)$ be two cycles.

Then $\alpha\beta = (1 2 5)(2 1 4 5 6)$

$$= \begin{pmatrix} 1 & 2 & 5 \\ 2 & 5 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 4 & 5 & 6 \\ 1 & 4 & 5 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 & 4 & 5 & 6 \\ 1 & 3 & 4 & 5 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix} \begin{pmatrix} 2 & 5 & 3 & 4 & 1 & 6 \\ 1 & 6 & 3 & 5 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 3 & 5 & 4 & 2 \end{pmatrix}, \quad \text{This is not a cycle.}$$

Thus product of two cycles need not be a cycle.

Very Important
Theorem

Every permutation of degree n can be expressed as a product of cyclic permutations acting on mutually disjoint sets.

Proof:

Let α be permutation on X with n elements.

Suppose $a_1, a_2, \dots, a_k \in X$, on which α acts as

$$\begin{array}{ll}
 a_1 \rightarrow a_2 & (a_1) \alpha = a_2 \\
 a_2 \rightarrow a_3 & (a_2) \alpha = a_3 \\
 a_3 \rightarrow a_4 & \dots\dots\dots \\
 \vdots & \dots\dots\dots \\
 a_k \rightarrow a_1 & (a_k) \alpha = a_1
 \end{array}$$

$$\text{Let } \alpha_1 = \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ a_2 & a_3 & \dots & a_1 \end{pmatrix}, \quad \text{where } k \leq n$$

If $k = n$ then $\alpha = \alpha_1$ Hence the proof.

If $k < n$ then let $b_1, b_2, \dots, b_p \in X - \{a_1, a_2, \dots, a_k\}$ on which α acts as

$$\begin{array}{ll}
 (b_1) \alpha = b_2 \\
 (b_2) \alpha = b_3 \\
 \dots\dots\dots \\
 (b_p) \alpha = b_1
 \end{array}$$

$$\text{Let } \alpha_2 = \begin{pmatrix} b_1 & b_2 & \dots & b_p \\ b_2 & b_3 & \dots & b_1 \end{pmatrix}, \quad \text{where } k + p \leq n$$

If $k + p = n$ then $\alpha = \alpha_1 \alpha_2$ Hence the proof.

If $k + p < n$ then we continue this process of extracting each time a cycle. As degree of α is finite therefore this process must end at some cycle say α_q where

$$\alpha_q = \begin{pmatrix} c_1 & c_2 & \dots & c_q \\ c_2 & c_3 & \dots & c_1 \end{pmatrix}$$

$$\text{Hence } \alpha = \alpha_1 \alpha_2 \dots \alpha_q \quad \text{where } k + p + \dots + q = n$$

Where $\{a_1, a_2, \dots, a_k\} \cap \{b_1, b_2, \dots, b_p\} \cap \dots \cap \{c_1, c_2, \dots, c_q\} = \phi$

Otherwise α will not be a bijective function.

In fact

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_k & b_1 & b_2 & \dots & b_p & c_1 & c_2 & \dots & c_q \\ a_2 & a_3 & \dots & a_1 & b_2 & b_3 & \dots & b_1 & c_2 & c_3 & \dots & c_1 \end{pmatrix}$$

TRANSPOSITIONS

Transposition

A cycle of length 2 is called a transposition. Thus the permutation of the type $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is called a transposition.

Theorem *Important*

Prove that every cyclic permutation can be expressed as a product of transpositions.

Proof:

Let α be a cyclic permutation of length n . i.e. $\alpha = (a_1 a_2 \dots a_n)$

We shall prove that

$$(a_1 a_2 \dots a_n) = (a_1 a_2)(a_1 a_3) \dots (a_1 a_n) \quad \dots \dots (1)$$

We shall prove (1) by principle of mathematical induction.

For $n = 2$

$$\text{L.H.S} = (a_1 a_2)$$

$$\text{R.H.S} = (a_1 a_2)$$

Hence (1) is true for $n = 2$

For $n = 3$

$$\text{L.H.S of (1)} = (a_1 a_2 a_3)$$

$$\text{R.H.S of (1)} = (a_1 a_2)(a_1 a_3)$$

$$= \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} a_1 & a_3 \\ a_3 & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_2 & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_3 \end{pmatrix} \begin{pmatrix} a_2 & a_1 & a_3 \\ a_2 & a_3 & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{pmatrix}$$

$$= (a_1 a_2 a_3)$$

$$= \text{L.H.S of (1)}$$

For $n = k$

Suppose (1) is true for k

$$\text{i.e. } (a_1 a_2 \dots a_k) = (a_1 a_2)(a_1 a_3) \dots (a_1 a_k) \quad \dots \dots (2)$$

For $n = k + 1$

We shall prove that (1) is true for $n = k + 1$

$$\text{i.e. } (a_1 a_2 \dots a_{k+1}) = (a_1 a_2)(a_1 a_3) \dots (a_1 a_{k+1})$$

$$\begin{aligned}
\text{R.H.S} &= (a_1 a_2)(a_1 a_3) \dots \dots \dots (a_1 a_k)(a_1 a_{k+1}) \\
&= (a_1 a_2 \dots \dots \dots a_k)(a_1 a_{k+1}) \quad \text{By (2)} \\
&= \begin{pmatrix} a_1 & a_2 & \dots & \dots & a_k \\ a_2 & a_3 & \dots & \dots & a_1 \end{pmatrix} \begin{pmatrix} a_1 & a_{k+1} \\ a_{k+1} & a_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & a_2 & \dots & \dots & a_k & a_{k+1} \\ a_2 & a_3 & \dots & \dots & a_1 & a_{k+1} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \dots & \dots & a_k & a_{k+1} \\ a_{k+1} & a_2 & \dots & \dots & a_k & a_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & a_2 & \dots & \dots & a_k & a_{k+1} \\ a_2 & a_3 & \dots & \dots & a_1 & a_{k+1} \end{pmatrix} \begin{pmatrix} a_2 & a_3 & \dots & \dots & a_1 & a_{k+1} \\ a_2 & a_3 & \dots & \dots & a_{k+1} & a_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & a_2 & \dots & \dots & a_k & a_{k+1} \\ a_2 & a_3 & \dots & \dots & a_{k+1} & a_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \begin{pmatrix} a_1 & a_3 \\ a_3 & a_4 \end{pmatrix} \dots \dots \dots \begin{pmatrix} a_1 & a_{k+1} \\ a_{k+1} & a_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & a_2 & \dots & \dots & a_{k+1} \\ a_2 & a_3 & \dots & \dots & a_1 \end{pmatrix} \\
&= \text{L.H.S}
\end{aligned}$$

Theorem

Every permutation of degree n can be expressed as a product of transpositions.

Proof:

Let α be a permutation of degree n . We know that

$$\alpha = \text{Product of disjoint cycles} \dots \dots \dots (1)$$

($\because$ Every permutation can be expressed as a product of disjoint cycles.)

Also, A cycle = Product of transpositions $\dots \dots \dots (2)$

($\because$ Every cyclic permutation can be expressed as a product of transpositions.)

From (1) and (2), It is clear that

$$\alpha = \text{Product of transposition}$$

Even Permutation

A permutation α in S_n is said to be an even permutation if it can be written as a product of an even number of transpositions.

For example

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} = (1 \ 2)(3 \ 4)$$

Clearly the number of transpositions in the decomposition of α is 2, which is an even number so, α is an even permutation.

Odd Permutation

A permutation β in S_n is said to be an odd permutation if it can be written as a product of an odd number of transpositions.

For example

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1 \ 2 \ 3 \ 4) = (1 \ 2)(1 \ 3)(1 \ 4)$$

Clearly the number of transpositions in the decomposition of α is 3, which is an odd number so, β is an odd permutation.

Theorem

- (i) The product of two even or two odd permutations is an even permutation.
- (ii) The product of an even permutation and an odd permutation is an odd permutation.

Proof:

- (i) (a) Let α and β are two even permutations. Then α and β can be written as a product of even number of transpositions. Let α can be written as a product of $2m$ transpositions and β can be written as a product of $2n$ transpositions.

Then the product $\alpha\beta$ can be written as a product of $2m + 2n = 2(m + n)$ transpositions i.e. even number of transpositions [$\because 2(m + n)$ is an even number].

Thus the product of two even permutations is an even permutation.

- (b) Let f and g are two odd permutations. Then f and g can be written as a product of odd number of transpositions. Let f can be written as a product of $2m + 1$ transpositions and g can be written as a product of $2n + 1$ transpositions.

Then the product fg can be written as a product of $(2m + 1) + (2n + 1) = 2(m + n + 1)$ transpositions i.e. even number of transpositions [$\because 2(m + n + 1)$ is an even number].

Thus the product of two odd permutations is an even permutation.

- (ii) Let α be an even permutation and β be an odd permutation. Then α and β can be written as a product of an even and an odd number of transpositions respectively. Let α can be written as a product of $2m$ transpositions and β can be written as a product of $2n + 1$ transpositions.

Then the product $\alpha\beta$ can be written as a product of $2m + (2n + 1) = 2(m + n) + 1$ transpositions i.e. odd number of transpositions [$\because 2(m + n) + 1$ is an odd number].

Thus the product of an even and an odd permutation is an odd permutation.

Theorem

Let α be any permutation of degree n and τ be a transposition then $\alpha\tau$ or $\tau\alpha$ is an even or an odd permutation according as α is odd or even respectively.

Proof:

Suppose α is an even permutation.

Then α = Product of m transpositions. (Where m is even.)

$\Rightarrow \tau\alpha$ = Product of $(m + 1)$ transpositions. (Where $(m + 1)$ is odd.)

$\Rightarrow \tau\alpha$ is an odd permutation.

Suppose α is an odd permutation.

Then $\alpha = \text{Product of } n \text{ transpositions.}$ (Where n is odd.)

$\Rightarrow \tau\alpha = \text{Product of } (n+1) \text{ transpositions.}$ (Where $(n+1)$ is even.)

$\Rightarrow \tau\alpha$ is an even permutation.

Theorem

For $n \geq 2$, the number of even permutations in S_n is equal to the number of odd permutations in S_n .

Proof:

We know that S_n is called symmetric group of degree n .

Also the order of $S_n = n!$

Let $p_1, p_2, \dots, p_r$ be even permutations,

And $q_1, q_2, \dots, q_s$ be odd permutations.

Where $r + s = n!$

Let τ be a transposition.

Then $\tau p_1, \tau p_2, \dots, \tau p_r$ are all distinct odd permutations.

[$\because$ If $\tau p_i = \tau p_j$ for $i \neq j$ then $p_i = p_j$ (cancellation law) $\Rightarrow i = j$ a contradiction]

Thus we have " r " odd permutations in S_n .

But number of odd permutations in S_n is " s "

$$\therefore s \geq r \quad \dots \dots \dots (1)$$

Now $\tau q_1, \tau q_2, \dots, \tau q_s$ are all distinct even permutations.

Thus we have " s " even permutations in S_n .

But number of even permutations in S_n is " r "

$$\therefore r \geq s \quad \dots \dots \dots (2)$$

From (1) and (2) it is clear that $r = s$

$$\text{Since } r + s = n! \Rightarrow r = s = \frac{n!}{2}$$

Theorem

Prove that the set A_n of all even permutations in S_n form a subgroup of S_n . Explain why the set B_n of all odd permutations in S_n do not form a subgroup of S_n ?

Proof:

Let A_n be the set of all even permutations in S_n .

We shall prove that A_n is a subgroup of S_n .

Obviously $A_n \subseteq S_n$

Let $\alpha, \beta \in A_n$. We have to prove that $\alpha\beta^{-1} \in A_n$.

Now α and β are even permutations.

Also β^{-1} is an even permutation.

($\because$ Inverse of an even permutation is an even permutation)

Hence $\alpha\beta^{-1}$ is an even permutation.

($\because$ Product of two even permutations is an even permutation)

Thus $\alpha\beta^{-1} \in A_n$

Since $\alpha, \beta \in A_n \Rightarrow \alpha\beta^{-1} \in A_n, \forall \alpha, \beta \in A_n$

Thus A_n is a subgroup of S_n .

Even $\times$ Even = Even
odd $\times$ odd = Even.

Let B_n be the set of all odd permutations in S_n .

Obviously $B_n \subseteq S_n$

Let $f, g \in B_n$.

Now f and g are odd permutations.

Also g^{-1} is an odd permutation.

($\because$ Inverse of an odd permutation is an odd permutation)

But fg^{-1} is not an odd permutation.

($\because$ Product of two odd permutations is an even permutation)

Thus $fg^{-1} \notin B_n$

Since $f, g \in B_n \nRightarrow fg^{-1} \in B_n, \forall f, g \in B_n$

Thus B_n is not a subgroup of S_n .

Example ✓

Find all the subgroup of S_3

Solution

Here S_3 is set of all permutations defined on a set X with three elements also called symmetric group of degree 3.

Let $X = \{1, 2, 3\}$. Then all the permutations on X are

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \alpha\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \alpha^2\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Thus $S_3 = \{I, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta\}$

$$\text{Where } \alpha^3 = \beta^2 = (\alpha\beta)^2 = (\alpha^2\beta)^2 = I$$

Trivial subgroups of S_3 are I, S_3

Non-trivial subgroups of S_3 are

$$\text{Let } H_1 = \langle \alpha : \alpha^3 = I \rangle = \{\alpha, \alpha^2, \alpha^3 = I\} = \{\alpha, \alpha^2, I\}$$

$$H_2 = \langle \beta : \beta^2 = I \rangle = \{\beta, \beta^2 = I\} = \{\beta, I\}$$

$$H_3 = \langle \alpha\beta : (\alpha\beta)^2 = I \rangle = \{\alpha\beta, I\}$$

$$H_4 = \langle \alpha^2\beta : (\alpha^2\beta)^2 = I \rangle = \{\alpha^2\beta, I\}$$

Order of a permutation

Let α be a permutation. Then the least positive integer m is called order of α if $\alpha^m = I$, where I is an identity permutation.

$$\text{Let } \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\text{Then } \alpha^2 = \alpha \cdot \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\begin{aligned}
 \text{Also } \alpha^3 &= \alpha \cdot \alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I
 \end{aligned}$$

Since $\alpha^3 = I \Rightarrow$ Order of $\alpha = 3$

Note

The order of a transposition is 2.

Let $\tau = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ be a transposition.

$$\begin{aligned}
 \text{Then } \tau^2 &= \tau \cdot \tau = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} \\
 &= \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} b & a \\ a & b \end{pmatrix} \\
 &= \begin{pmatrix} a & b \\ a & b \end{pmatrix} \\
 &= I
 \end{aligned}$$

Since $\tau^2 = I \Rightarrow$ Order of $\tau = 2$

Theorem

The order of cyclic permutation of length m is m .

Proof:

Let α be a cyclic permutation of length m .

$$\text{i.e. } \alpha = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix}$$

$$\begin{aligned}
 \text{Then } \alpha^2 &= \alpha \cdot \alpha = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix} \\
 &= \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix} \begin{pmatrix} a_2 & a_3 & a_4 & \dots & a_1 \\ a_3 & a_4 & a_5 & \dots & a_2 \end{pmatrix} \\
 &= \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_3 & a_4 & a_5 & \dots & a_2 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \alpha^3 &= \alpha^2 \cdot \alpha = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_3 & a_4 & a_5 & \dots & a_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix} \\
 &= \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_3 & a_4 & a_5 & \dots & a_2 \end{pmatrix} \begin{pmatrix} a_3 & a_4 & a_5 & \dots & a_2 \\ a_4 & a_5 & a_6 & \dots & a_3 \end{pmatrix} \\
 &= \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_4 & a_5 & a_6 & \dots & a_3 \end{pmatrix}
 \end{aligned}$$

$$\text{Similarly } \alpha^4 = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_5 & a_6 & a_7 & \dots & a_4 \end{pmatrix}$$

$$\text{Also } \alpha^m = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_1 & a_2 & a_3 & \dots & a_m \end{pmatrix}$$

Thus Order of $\alpha = m =$ Length of α

Theorem

The order of a permutation is the least common multiple (L.C.M) of the orders of the disjoint cycles into whose product it is decomposed.

Proof:

Let α be any permutation of degree n .

We know that α can be written as a product of disjoint cycles say

$$\alpha_1, \alpha_2, \dots, \alpha_k$$

$$\text{Then } \alpha = \alpha_1 \alpha_2 \dots \alpha_k \quad (1)$$

Suppose $m_i = \text{Length of } \alpha_i$ where $i = 1, 2, \dots, k$

$$\Rightarrow m_i = \text{Length of } \alpha_i \text{ where } i = 1, 2, \dots, k$$

($\because$ Length of a cycle = Order of a cycle)

Let L.C.M of $m_i = m$, $i = 1, 2, \dots, k$

Then $m = m_i q_i$ where $i = 1, 2, \dots, k$, and $q_i \in \mathbb{Z}$

Now from equation (1)

$$\begin{aligned} \alpha^m &= \alpha_1^m \alpha_2^m \dots \alpha_k^m \\ &= \alpha_1^{m_1 q_1} \alpha_2^{m_2 q_2} \dots \alpha_k^{m_k q_k} \\ &= (\alpha_1^{m_1})^{q_1} (\alpha_2^{m_2})^{q_2} \dots (\alpha_k^{m_k})^{q_k} \\ &= (I)^{q_1} (I)^{q_2} \dots (I)^{q_k} \quad \because m_i = O(\alpha_i) \\ &= I \cdot I \dots I \\ &= I \end{aligned}$$

$$12 = \text{L.C.M of } (2, 3, 4)$$

$$\therefore 12 = 2q, q = 6$$

$$12 = 3q, q = 4$$

$$12 = 4q, q = 3$$

Hence order of $\alpha = m$

$$= \text{L.C.M of } m_1, m_2, \dots, m_k$$

$$= \text{L.C.M of } O(\alpha_1), O(\alpha_2), \dots, O(\alpha_k)$$

Example

Find the order of $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 7 & 9 & 6 & 5 & 8 & 12 & 11 & 10 \end{pmatrix}$

Solution

$$\begin{aligned} \text{Here } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 7 & 9 & 6 & 5 & 8 & 12 & 11 & 10 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 5 & 7 & 6 & 9 & 8 \\ 7 & 6 & 9 & 8 & 5 \end{pmatrix} \begin{pmatrix} 11 \\ 11 \end{pmatrix} \begin{pmatrix} 10 & 12 \\ 12 & 10 \end{pmatrix} \\ &= (1 \ 2 \ 3 \ 4)(5 \ 7 \ 6 \ 9 \ 8)(11)(10 \ 12) \end{aligned}$$

$$\text{Let } \alpha_1 = (1 \ 2 \ 3 \ 4), \alpha_2 = (5 \ 7 \ 6 \ 9 \ 8), \alpha_3 = (11), \alpha_4 = (10 \ 12)$$

Then lengths of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are 4, 5, 1, 2

$\therefore$ Orders of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are 4, 5, 1, 2

Thus order of $\alpha = \text{L.C.M of } [O(\alpha_1), O(\alpha_2), O(\alpha_3), O(\alpha_4)]$

$$= \text{L.C.M of } [4, 5, 1, 2]$$

$$= 20$$

EXERCISE 2.3

Q. No.1

Multiply the following permutations.

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 4 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 1 & 4 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 9 & 2 & 3 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 3 & 9 & 2 \end{pmatrix}$

Solution

(i) Here $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 & 4 & 1 \\ 1 & 4 & 5 & 2 & 3 \end{pmatrix}$
 $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 2 & 3 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 & 6 & 4 & 1 \\ 3 & 2 & 1 & 4 & 5 & 6 \end{pmatrix}$
 $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 4 & 5 & 6 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 9 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 3 & 9 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 9 & 2 & 3 \end{pmatrix} \begin{pmatrix} 8 & 6 & 7 & 9 & 2 & 3 \\ 9 & 7 & 3 & 2 & 8 & 6 \end{pmatrix}$
 $= \begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 9 & 7 & 3 & 2 & 8 & 6 \end{pmatrix}$

Q. No.2

Find the inverse of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$

Solution

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$

Then $\alpha^{-1} = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

Q. No.3

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 4 & 1 & 7 & 2 & 6 \end{pmatrix}$, find the inverse of α .

Solution

Here $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 4 & 1 & 7 & 2 & 6 \end{pmatrix}$

Then $\alpha^{-1} = \begin{pmatrix} 3 & 5 & 4 & 1 & 7 & 2 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$

Q. No.4

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

Show that $f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$, $f^2 \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

$f^3 \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$, Also $f^4 = g^2 = (f \circ g)^2$

$$\begin{aligned}
 \text{Now } h \circ g &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 5 & 4 & 1 & 2 & 6 & 3 \\ 2 & 1 & 6 & 5 & 3 & 4 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 3 & 4 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{And } h^2 &= h \circ h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 5 & 4 & 1 & 2 & 6 & 3 \\ 6 & 2 & 5 & 4 & 3 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 5 & 4 & 3 & 1 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{Also } g^2 &= g \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 6 & 5 & 4 & 1 & 2 & 3 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}
 \end{aligned}$$

Now we shall find the orders of f, g & h

$$\begin{aligned}
 \text{Here } f &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 4 & 5 & 3 & 6 \\ 2 & 4 & 5 & 3 & 6 & 1 \end{pmatrix}, \quad \text{This is a cycle of length 6.}
 \end{aligned}$$

$$\therefore \text{ Order of } f = 6 \quad (\because \text{Length of a cycle} = \text{Order of cycle})$$

$$\begin{aligned}
 \text{And } g &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 6 & 3 & 4 \\ 6 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} = (1 \ 6 \ 3 \ 4)(2 \ 5)
 \end{aligned}$$

$$\text{Let } \alpha_1 = (1 \ 6 \ 3 \ 4), \text{ and } \alpha_2 = (2 \ 5)$$

Then lengths of α_1, α_2 are 4, 2

$$\therefore \text{ Orders of } \alpha_1, \alpha_2 \text{ are } 4, 2$$

$$\begin{aligned}
 \text{Thus order of } g &= L.C.M \text{ of } [O(\alpha_1), O(\alpha_2)] \\
 &= L.C.M \text{ of } [4, 2] \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \text{Also } h &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 5 & 6 & 3 \\ 5 & 6 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} = (1 \ 5 \ 6 \ 3)(2 \ 4)
 \end{aligned}$$

$$\text{Let } \beta_1 = (1 \ 5 \ 6 \ 3), \text{ and } \beta_2 = (2 \ 4)$$

Then lengths of β_1, β_2 are 4, 2

$$\therefore \text{ Orders of } \beta_1, \beta_2 \text{ are } 4, 2$$

$$\begin{aligned}
 \text{Thus order of } h &= L.C.M \text{ of } [O(\beta_1), O(\beta_2)] \\
 &= L.C.M \text{ of } [4, 2] \\
 &= 4
 \end{aligned}$$

$$= \text{L.C.M of } [4, 2]$$

$$= 4$$

Q. No.6

Given the permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$, verify that $\alpha^6 = I$

Solution

$$\text{Here } \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 3 \end{pmatrix} = (1\ 2)(3\ 4\ 5)$$

Let $\alpha_1 = (1\ 2)$, and $\alpha_2 = (3\ 4\ 5)$

Then lengths of α_1, α_2 are 2, 3

$\therefore$ Orders of α_1, α_2 are 2, 3

$$\text{Thus order of } \alpha = \text{L.C.M of } [O(\alpha_1), O(\alpha_2)]$$

$$= \text{L.C.M of } [2, 3]$$

$$= 6$$

$$\Rightarrow \alpha^6 = I \quad (\text{where } I \text{ is an identity permutation.})$$

Q. No.7

Write the multiplication table for the permutations
(1), (1 2 3 4), (1 4 3 2), (1 3)(2 4).

Solution

$$\text{Let } \alpha_0 = (1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\alpha_1 = (1\ 2\ 3\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\alpha_2 = (1\ 4\ 3\ 2) = \begin{pmatrix} 1 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\alpha_3 = (1\ 3)(2\ 4) = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

We see that

$$\alpha_0 \alpha_0 = \alpha_0$$

$$\alpha_0 \alpha_1 = \alpha_1$$

$$\alpha_0 \alpha_2 = \alpha_2$$

$$\alpha_0 \alpha_3 = \alpha_3$$

$$\alpha_1 \alpha_0 = \alpha_1$$

$\circ$	α_0	α_1	α_2	α_3
α_0	α_0	α_1	α_2	α_3
α_1	α_1	α_3	α_0	α_2
α_2	α_2	α_0	α_3	α_1
α_3	α_3	α_2	α_1	α_0

$$\alpha_1 \alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \alpha_3$$

$$\alpha_1 \alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \alpha_0$$

$$\alpha_1 \alpha_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \alpha_2$$

$$\alpha_2 \alpha_0 = \alpha_2$$

$$= \{ \alpha, \alpha^2, \alpha^3, I \}$$

Q. No.9

Find the following products and express them as a product of cyclic permutations on mutually disjoint sets

$$(124)(13654), (1234)(52341), (14)(235)(35)(45)$$

Solution

$$\begin{aligned} \text{Here } (124)(13654) &= \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 6 & 5 & 4 \\ 3 & 6 & 5 & 4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 6 & 1 & 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix} \begin{pmatrix} 2 & 4 & 3 & 1 & 5 & 6 \\ 2 & 1 & 6 & 3 & 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 3 & 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 6 & 5 & 4 \\ 6 & 5 & 4 & 3 \end{pmatrix} \\ &= (12)(3654) \end{aligned}$$

$$\begin{aligned} \text{And } (1234)(25341) &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 & 3 & 4 & 1 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 5 & 3 & 4 & 1 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 1 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 5 & 3 & 2 & 4 \\ 5 & 3 & 1 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 5 & 3 \\ 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \\ &= (153)(24) \end{aligned}$$

$$\begin{aligned} \text{Now } (14)(235) &= \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 \\ 3 & 5 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 & 1 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \& \quad (14)(235)(35) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 3 & 5 & 1 & 2 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{So, } (14)(235)(35)(45) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 5 & 3 & 1 & 2 \\ 5 & 4 & 3 & 1 & 2 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = (1 \ 2 \ 3 \ 4)(3)$$

Q. No.10

Express the following permutations as a product of disjoint cycles

$$(i) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$$

$$(ii) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$$

Solution

$$\begin{aligned} \text{Here } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 8 & 2 & 3 & 6 & 4 & 5 & 7 \\ 8 & 1 & 2 & 6 & 4 & 3 & 7 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 8 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 6 & 4 \\ 6 & 4 & 3 \end{pmatrix} \begin{pmatrix} 5 & 7 \\ 7 & 5 \end{pmatrix} \\ &= (1 \ 8)(2)(3 \ 6 \ 4)(5 \ 7) \end{aligned}$$

$$\begin{aligned} \text{And } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix} &= \begin{pmatrix} 1 & 3 & 4 & 2 & 6 & 5 & 8 & 7 \\ 3 & 4 & 1 & 6 & 2 & 8 & 7 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 & 4 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} 5 & 8 & 7 \\ 8 & 7 & 5 \end{pmatrix} \\ &= (1 \ 3 \ 4)(2 \ 6)(5 \ 8 \ 7) \end{aligned}$$

Q. No.11

Express the following permutations as a product of transpositions

$$(i) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 1 & 6 & 5 & 3 & 8 & 9 & 7 \end{pmatrix}$$

$$(ii) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 5 & 6 & 7 & 8 \end{pmatrix}$$

Solution

$$\begin{aligned} \text{Here } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 1 & 6 & 5 & 3 & 8 & 9 & 7 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 4 & 6 & 3 \\ 2 & 4 & 6 & 3 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} \begin{pmatrix} 7 & 8 & 9 \\ 8 & 9 & 7 \end{pmatrix} \\ &= (1 \ 2 \ 4 \ 6 \ 3)(5)(7 \ 8 \ 9) \\ &= (1 \ 2)(1 \ 4)(1 \ 6)(1 \ 3)(7 \ 8)(7 \ 9) \end{aligned}$$

$$\begin{aligned} \text{And } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 5 & 6 & 7 & 8 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} \begin{pmatrix} 7 \\ 7 \end{pmatrix} \begin{pmatrix} 8 \\ 8 \end{pmatrix} \\ &= (1 \ 2 \ 3 \ 4)(5)(6)(7)(8) \\ &= (1 \ 2)(1 \ 3)(1 \ 4) \end{aligned}$$

Q. No.12Write (i) all even permutations in S_3 (ii) all odd permutations in S_3 **Solution**

We know that S_3 is the set of all permutations defined on a set with 3 elements.

$$\text{Let } X = \{1, 2, 3\}$$

Then all the permutations defined on the set X are

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = (1)(2)(3)$$

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3) = (1\ 2)(1\ 3)$$

$$f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} = (1\ 3\ 2) = (1\ 3)(1\ 2)$$

$$f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1\ 2)(3) = (1\ 2)$$

$$f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = (1)(2\ 3) = (2\ 3)$$

$$f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = (1\ 3)(2) = (1\ 3)$$

We see that in the decomposition of identity permutation I , number of transpositions is zero, which is an even number.

Since the permutations I, f_1, f_2 can be written as a product of even number of transpositions therefore, I, f_1, f_2 are even permutations.

Since the permutations f_3, f_4, f_5 can be written as a product of odd number of transpositions therefore, f_3, f_4, f_5 are odd permutations.

Q. No. 13

Are the permutations

$(1\ 2\ 3)(2\ 5\ 6)(4\ 3\ 5\ 1)$ and $(1\ 4\ 7)(3\ 4\ 5)(8\ 7)(8\ 3\ 4\ 5)$ even or odd?

Solution

$$\text{Since } (1\ 2\ 3)(2\ 5\ 6)(4\ 3\ 5\ 1) = (1\ 2)(1\ 3)(2\ 5)(2\ 6)(4\ 3)(4\ 5)(4\ 1) \\ \text{(Odd number of transpositions)}$$

$\therefore (1\ 2\ 3)(2\ 5\ 6)(4\ 3\ 5\ 1)$ is an odd permutation.

$$\& (1\ 4\ 7)(3\ 4\ 5)(8\ 7)(8\ 3\ 4\ 5) = (1\ 4)(1\ 7)(3\ 4)(3\ 5)(8\ 7)(8\ 3)(8\ 4)(8\ 5) \\ \text{(Even number of transpositions)}$$

$\therefore (1\ 4\ 7)(3\ 4\ 5)(8\ 7)(8\ 3\ 4\ 5)$ is an even permutation.

Q. No. 14

Find the orders of each of the following permutations.

$$(i) \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad (ii) \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

$$(iii) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix} \quad (iv) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$$

Solution

$$(i) \quad \text{Let } \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} = (1 \ 2)(3 \ 4)$$

$$\text{Suppose } \alpha_1 = (1 \ 2) \text{ and } \alpha_2 = (3 \ 4)$$

Then lengths of, α_1 and α_2 are 2, 2 respectively.

$\therefore$ Orders of α_1 and α_2 are 2, 2 respectively.

($\because$ Length of cycle = Order of cycle)

Thus order of $\alpha = L.C.M$ of $[O(\alpha_1), O(\alpha_2)]$

$$= L.C.M \text{ of } [2, 2]$$

$$= 2$$

$$(ii) \quad \text{Let } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = (1 \ 4 \ 3)(2)$$

$$\text{Suppose } \beta_1 = (1 \ 4 \ 3)$$

Then length of β_1 is 3

$\therefore$ Order of β_1 is 3

($\because$ Length of cycle = Order of cycle)

Thus order of $\beta = \text{order of } \beta_1$

$$= 3$$

$$(iii) \quad \text{Let } \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = (1 \ 4 \ 5)(2 \ 3)$$

$$\text{Suppose } \gamma_1 = (1 \ 4 \ 5) \text{ and } \gamma_2 = (2 \ 3)$$

Then lengths of γ_1 and γ_2 are 3, 2 respectively.

$\therefore$ Orders of γ_1 and γ_2 are 3, 2 respectively.

($\because$ Length of cycle = Order of cycle)

Thus order of $\gamma = L.C.M$ of $[O(\gamma_1), O(\gamma_2)]$

$$= L.C.M \text{ of } [3, 2]$$

$$= 6$$

$$(iv) \quad \text{Let } \delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \end{pmatrix} = (1 \ 2 \ 3)(4 \ 5 \ 6)$$

$$\text{Suppose } \delta_1 = (1 \ 2 \ 3) \text{ and } \delta_2 = (4 \ 5 \ 6)$$

Then lengths of δ_1 and δ_2 are 3, 3 respectively.

$\therefore$ Orders of δ_1 and δ_2 are 3, 3 respectively.

Thus order of $\delta = L.C.M$ of $[O(\delta_1), O(\delta_2)]$

$$= \text{L.C.M of } [3, 3]$$

$$= 3$$

Q. No.15

Determine whether the given permutations are even or odd.

$$(i) \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad (ii) \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}$$

$$(iii) \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 1 & 6 & 4 \end{pmatrix} \quad (iv) \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 2 & 6 & 7 & 5 \end{pmatrix}$$

Solution

$$(i) \quad \text{Let } \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} = (1 \ 2)(3 \ 4)$$

(Even number of transpositions)

Since α can be written as a product of even number of transpositions,

Therefore α is an even permutation.

$$(ii) \quad \text{Let } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} = (1 \ 5)(2 \ 3)(4)$$

(Even number of transpositions)

Since β can be written as a product of even number of transpositions,

Therefore β is an even permutation.

$$(iii) \quad \text{Let } \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 1 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 & 5 & 6 & 4 \\ 3 & 2 & 5 & 6 & 4 & 1 \end{pmatrix}$$

$$= (1 \ 3 \ 2 \ 5 \ 6 \ 4)$$

$$= (1 \ 3)(1 \ 2)(1 \ 5)(1 \ 6)(1 \ 4)$$

(Odd number of transpositions)

Since γ can be written as a product of even number of transpositions,

Therefore γ is an odd permutation.

$$(iv) \quad \text{Let } \delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 2 & 6 & 7 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 4 & 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 6 & 7 & 3 \end{pmatrix}$$

$$= (1 \ 4 \ 2 \ 3)(5 \ 6 \ 7)$$

$$= (1 \ 4)(1 \ 2)(1 \ 3)(5 \ 6)(5 \ 7)$$

(Odd number of transpositions)

Since δ can be written as a product of even number of transpositions,

Therefore δ is an odd permutation.