

Exercise 6.4

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Q1 (i) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $T(x_1, x_2, x_3) = (x_1, x_2, 0)$

Standard basis for $\mathbb{R}^3 = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$

$$T(1, 0, 0) = (1, 0, 0) = 1(1, 0, 0) + 0(0, 1, 0) + 0(0, 0, 1)$$

$$T(0, 1, 0) = (0, 1, 0) = 0(1, 0, 0) + 1(0, 1, 0) + 0(0, 0, 1)$$

$$T(0, 0, 1) = (0, 0, 0) = 0(1, 0, 0) + 0(0, 1, 0) + 0(0, 0, 1)$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) $T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

Standard basis for $\mathbb{R}^3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$$T(1, 0, 0) = (1, -1, 0) = 1(1, 0, 0) - 1(0, 1, 0) + 0(0, 0, 1)$$

$$T(0, 1, 0) = (1, -1, 0) = 1(1, 0, 0) - 1(0, 1, 0) + 0(0, 0, 1)$$

$$T(0, 0, 1) = (0, 0, 1) = 0(1, 0, 0) + 0(0, 1, 0) + 1(0, 0, 1)$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii)

$$T(x_1, x_2, x_3) = (x_2, -x_1, -x_3)$$

Standard basis for $\mathbb{R}^3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$$T(1, 0, 0) = (0, -1, 0) = 0(1, 0, 0) - 1(0, 1, 0) + 0(0, 0, 1)$$

$$T(0, 1, 0) = (1, 0, 0) = 1(1, 0, 0) + 0(0, 1, 0) + 0(0, 0, 1)$$

$$T(0, 0, 1) = (0, 0, -1) = 0(1, 0, 0) + 0(0, 1, 0) - 1(0, 0, 1)$$

Hence the standard matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(iv)

$$T(x_1, x_2, x_3) = (x_1, x_2 + x_3, x_1 + x_2 + x_3)$$

Standard basis for $\mathbb{R}^3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$$T(1, 0, 0) = (1, 0, 1) = 1(1, 0, 0) + 0(0, 1, 0) + 1(0, 0, 1)$$

$$T(0, 1, 0) = (0, 1, 1) = 0(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1)$$

$$T(0, 0, 1) = (0, 1, 1) = 0(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1)$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

2 (i) $T: \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $T(x) = (3x, 5x)$

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Standard basis for $\mathbb{R} = \{1\}$

Standard basis for $\mathbb{R}^2 = \{(1, 0), (0, 1)\}$

$$T(1) = (3, 5) = 3(1, 0) + 5(0, 1)$$

So the matrix of T is $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

(ii) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (3x_1 - 4x_2 + 9x_3, 5x_1 + 3x_2 - 2x_3)$

Standard basis for $\mathbb{R}^3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Standard basis for $\mathbb{R}^2 = \{(1, 0), (0, 1)\}$

$$T(1, 0, 0) = (3, 5) = 3(1, 0) + 5(0, 1)$$

$$T(0, 1, 0) = (-4, 3) = -4(1, 0) + 3(0, 1)$$

$$T(0, 0, 1) = (9, -2) = 9(1, 0) - 2(0, 1)$$

So the matrix of T is $\begin{bmatrix} 3 & -4 & 9 \\ 5 & 3 & -2 \end{bmatrix}$

(iii) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^4$ defined by

$$T(x_1, x_2) = (3x_1 + 4x_2, 5x_1 - 2x_2, x_1 + 7x_2, 4x_1)$$

Standard basis for $\mathbb{R}^2 = \{(1, 0), (0, 1)\}$

Standard basis for $\mathbb{R}^4 = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

$$T(1, 0) = (3, 5, 1, 4) = 3(1, 0, 0, 0) + 5(0, 1, 0, 0) + (0, 0, 1, 0) + 4(0, 0, 0, 1)$$

$$T(0, 1) = (4, -2, 7, 0) = 4(1, 0, 0, 0) - 2(0, 1, 0, 0) + 7(0, 0, 1, 0) + 0(0, 0, 0, 1)$$

So the matrix of T is $\begin{bmatrix} 3 & 4 \\ 5 & -2 \\ 1 & 7 \\ 4 & 0 \end{bmatrix}$.

(iv) $T: \mathbb{R}^4 \rightarrow \mathbb{R}$ defined by

$$T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_2 - 7x_3 + x_4$$

Standard basis for $\mathbb{R}^4 = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

Standard basis for $\mathbb{R} = \{1\}$

$$T(1, 0, 0, 0) = 2 = 2(1)$$

$$T(0, 1, 0, 0) = 3 = 3(1)$$

$$T(0, 0, 1, 0) = -7 = -7(1)$$

$$T(0, 0, 0, 1) = 1 = 1(1)$$

So the matrix of T is $\begin{bmatrix} 2 & 3 & -7 & 1 \end{bmatrix}$.

Q.3 (ii)

Let $A = \begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

Since matrix A is of order 3×2 , so $m=3$, $n=2$.

i.e. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is defined by

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 - x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

or $T(x_1, x_2) = (6x_1 - x_2, x_1 + 2x_2, x_1 + 3x_2)$.

(ii)

Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}$$

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Since matrix A is of order 3×3 , so $m=3$, $n=3$.

i.e., $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 5x_2 + 6x_3 \\ -2x_1 + 3x_2 - x_3 \end{bmatrix}$$

or $T(x_1, x_2, x_3) = (x_1 + x_2 + 2x_3, 2x_1 + 5x_2 + 6x_3, -2x_1 + 3x_2 - x_3)$

(iii)

Let

$$A = \begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}$$

Since matrix A is of order 3×5 , so $m=3$, $n=5$.

i.e., $T: \mathbb{R}^5 \rightarrow \mathbb{R}^3$ is defined by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_1 + x_2 + 2x_4 \\ x_1 + x_4 + x_5 \\ -x_2 + x_3 + x_4 + x_5 \end{bmatrix}$$

or

$T(x_1, x_2, x_3, x_4, x_5) = (3x_1 + x_2 + 2x_4, x_1 + x_4 + x_5, -x_2 + x_3 + x_4 + x_5).$

④

The matrix of $T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$

T in terms of coordinates is

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_1 - x_3 \\ -x_1 - x_2 \end{bmatrix}$$

or

$T(x_1, x_2, x_3) = (x_2 + x_3, x_1 - x_3, -x_1 - x_2)$

Given basis for $\mathbb{R}^3 = \{v_1 = (0, 1, 2), v_2 = (1, 1, 1), v_3 = (1, 0, -2)\}$

$$T(v_1) = T(0, 1, 2) = (3, -2, -1)$$

$$(3, -2, -1) = a_1 v_1 + a_2 v_2 + a_3 v_3$$

$$(3, -2, -1) = a_1(0, 1, 2) + a_2(1, 1, 1) + a_3(3, -2, -1)$$

$$(3, -2, -1) = (a_2 + a_3, a_1 + a_2, 2a_1 + a_2 - 2a_3)$$

$$a_2 + a_3 = 3$$

$$a_1 + a_2 = -2$$

$$2a_1 + a_2 - 2a_3 = -1$$

$$a_2 = 3 - a_3$$

$$a_1 + (3 - a_3) = -2$$

$$2(-5 + a_3) + (3 - a_3) - 2a_3 = -1$$

$$a_1 = -2 - 3 + a_3$$

$$-10 + 2a_3 + 3 - a_3 - 2a_3 = -1$$

$$a_1 = -5 + a_3$$

$$-a_3 - 7 = -1$$

$$-a_3 = 6$$

$$\boxed{a_3 = -6}$$

$$a_2 = 3 - (-6)$$

$$\boxed{a_2 = 9}$$

$$a_1 = -5 - 6$$

$$\boxed{a_1 = -11}$$

$$\text{So } T(v_1) = -11v_1 + 9v_2 - 6v_3 \quad - \textcircled{1}$$

$$\text{Similarly } T(v_2) = -2v_1 + 2v_2 + 0v_3 \quad - \textcircled{2}$$

$$\text{and } T(v_3) = 14v_1 - 11v_2 + 9v_3 \quad - \textcircled{3}$$

Hence the matrix of T w.r.t. new basis is

$$\begin{bmatrix} -11 & -2 & 14 \\ 9 & 2 & -11 \\ -6 & 0 & 9 \end{bmatrix}$$

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$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$$T(1, 1) = (0, 1, 2) \quad \text{and} \quad T(-1, 1) = (2, 1, 0)$$

Standard basis for $\mathbb{R}^2 = \{(1, 0), (0, 1)\}$

Standard basis for $\mathbb{R}^3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Since $(1, 1) = 1(1, 0) + 1(0, 1)$

and $(-1, 1) = -1(1, 0) + 1(0, 1)$

So $(0, 1, 2) = T(1, 1) = T(1, 0) + T(0, 1) \quad \text{--- ①}$

and $(2, 1, 0) = T(-1, 1) = -T(1, 0) + T(0, 1) \quad \text{--- ②}$

Subtract ② from ①, we get

$$(-2, 0, 2) = 2T(1, 0) + 0$$

$$\Rightarrow T(1, 0) = (-1, 0, 1) = -1(1, 0, 0) + 0(0, 1, 0) + 1(0, 0, 1)$$

Add ① and ② to get

$$(2, 2, 2) = 2T(0, 1)$$

$$\Rightarrow T(0, 1) = (1, 1, 1) = 1(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1)$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

⑥

 $T: M_{23} \rightarrow M_{32}$ defined by $T(A) = A^T \in M_{32}$

Standard basis for $M_{23} = \left\{ \overset{v_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{v_4}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}, \overset{v_5}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}, \overset{v_6}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \right\}$

Standard basis for $M_{32} = \left\{ w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, w_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, w_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, w_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$T(v_1) = w_2 = 0w_1 + w_2 + 0w_3 + 0w_4 + 0w_5 + 0w_6$$

$$T(v_2) = w_4 = 0w_1 + 0w_2 + 0w_3 + w_4 + 0w_5 + 0w_6$$

$$T(v_3) = w_6 = 0w_1 + 0w_2 + 0w_3 + 0w_4 + 0w_5 + w_6$$

$$T(v_4) = w_1 = w_1 + 0w_2 + 0w_3 + 0w_4 + 0w_5 + 0w_6$$

$$T(v_5) = w_3 = 0w_1 + 0w_2 + w_3 + 0w_4 + 0w_5 + 0w_6$$

$$T(v_6) = w_5 = 0w_1 + 0w_2 + 0w_3 + 0w_4 + w_5 + 0w_6$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

If we change the order of basis elements, then there will be different matrix of T .