

General Theorems, indeterminate forms)

(Chapter no. 3)

Before starting this chapter, we should know the following terms.

Definition ① Let S be a non empty subset of real no's. An element $M \in \mathbb{R}$ is said to be an upper bound of S if

$$x \leq M \quad \forall x \in S$$

An element $m \in \mathbb{R}$ is said to be a lower bound of S if

$$m \leq x \quad \forall x \in S$$

If S has an upper bound then S is said to be bounded above. If S has a lower bound then S is said to be bounded below.

A set S is said to be a bounded set if it is both bounded above and below.

Ex ①

$$\text{Let } S = \{1, 2, 3, \dots, 20\}$$

Then every real no. $M \geq 20$ is an upper bound of S and every real no. $m \leq 1$ is a lower bound of S .

$$\text{Ex ② Let } S = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$$

$$\text{An upper bound of } S = 1$$

$$\text{A lower bound of } S = 0$$

Def ② Let S be a non empty subset of $\mathbb{R}$ ②

① If S is bounded above then an upper bound of S is called a least upper bound of S or supremum of S if M is less than any other upper bound of S .

② If S is bounded below then a lower bound m of S is called greatest lower bound of S or infimum of S if m is greater than any other lower bound of S . In notation we write it as

$$M = \sup S \quad \text{or} \quad M = \text{lub } S$$

$$m = \inf S \quad \text{or} \quad m = \text{glb } S$$

In example ①

$$M = 2.0$$

$$m = 1$$

In example ②

$$M = 1, \quad m = 0$$

Def ③ A function f is said to be continuous in $[a, b]$ if there is no gap, or breakage in the graph of f in the interval $[a, b]$.

e.g., In fig f is continuous in $[1, 2]$ & discontinuous in $[4, 5]$



Rolle's theorem :-statement:let a function f be

- ① Continuous on closed interval $[a, b]$.
- ② Derivable in the open interval $]a, b[$
- ③ $f(a) = f(b)$

Then there exists at least one point $c \in]a, b[$ such that $f'(c) = 0$

proof:-

Because f is continuous on $[a, b]$ & so it is bounded.

$$\text{let } M = \sup f \\ \text{ \& } m = \inf f$$

Case ① If $M = m$ Then f is const. on $[a, b]$ & so

$$f'(x) = 0 \quad \forall x \in]a, b[$$

& we have the required proof.

Case ② If $M \neq m$

Then at least one of M & m is different from $f(a)$ & $f(b)$.

$$\text{Suppose } M \neq f(a) = f(b) \quad \text{--- ①}$$

Since f attains its supremum on $[a, b]$, so there is a pt. $c \in [a, b]$ s.t. $f(c) = M$

But from ① $M \neq f(a) = f(b)$

$$\text{so } c \neq a \text{ \& } c \neq b$$

$$\Rightarrow c \in]a, b[$$

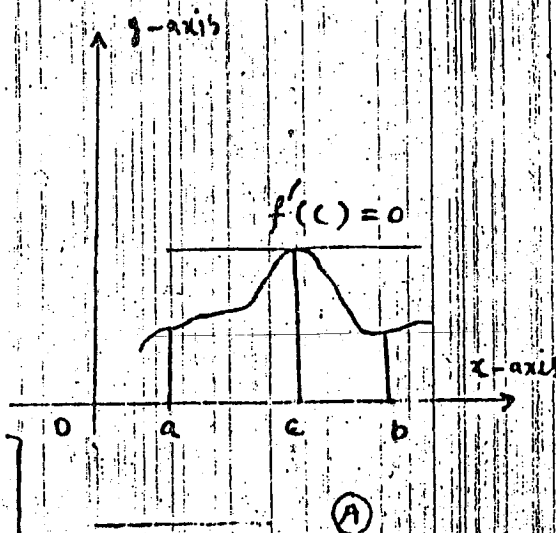
Suppose that h be a +ve real no. then

$$\left. \begin{aligned} f(c+h) &\leq f(c) \\ f(c-h) &\leq f(c) \end{aligned} \right\}$$

$$\left. \begin{aligned} \frac{f(c+h) - f(c)}{h} &\leq 0 \\ \frac{f(c-h) - f(c)}{h} &\leq 0 \end{aligned} \right\}$$

or

$$\left. \begin{aligned} \frac{f(c+h) - f(c)}{h} &\leq 0 \\ -\frac{f(c-h) - f(c)}{-h} &\geq 0 \end{aligned} \right\}$$



Taking limit as $h \rightarrow 0$

Eq. (A)

becomes

$$\left. \begin{aligned} f'(c) &\leq 0 \\ f'(c) &\geq 0 \end{aligned} \right\}$$

$$\Rightarrow \boxed{f'(c) = 0} \quad \text{--- As required}$$

Mean-Value theorem (OR) (Lagrange's M.V.T.) :-

Statement:-

Let a function f be:

- ① continuous on closed interval $[a, b]$
- ② Derivable in the open interval $]a, b[$

Then there exists a point $c \in]a, b[$ s. that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Proof:-

Define a new function

$$\phi(x) = Ax + f(x) \quad \text{--- (1)}$$

where A is a const. to be determined s. that

$$\phi(a) = \phi(b)$$

obviously the function Ax is continuous on $[a, b]$ & derivable in $]a, b[$.

Now $\phi(x)$ satisfies all the conditions of Rolle's theorem. So there is a point:

$$c \in]a, b[, \text{ s. that}$$

$$\phi'(c) = 0$$

$$A + f'(c) = 0$$

$$\Rightarrow f'(c) = -A \quad \text{--- (2)}$$

$$\text{From } \phi(a) = \phi(b)$$

$$\Rightarrow Aa + f(a) = Ab + f(b) \quad \text{or}$$

$$Aa - Ab = f(b) - f(a)$$

$$A(a - b) = f(b) - f(a)$$

$$\Rightarrow -A(b - a) = f(b) - f(a)$$

$$= -A = \frac{f(b) - f(a)}{b - a}$$

Put value of $-A$ in (2)

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

So $\frac{f(b) - f(a)}{b - a} = f'(c)$ as desired.

Note :- Another form of mean value theorem

If we take the interval as $[a, a+h]$ instead of $[a, b]$. Then mean value theorem becomes :-

$$\frac{f(a+h) - f(a)}{a+h-a} = f'(c)$$

where c is a pt s. that $a < c < a+h$

$$\Rightarrow f(a+h) - f(a) = h f'(c) \quad \text{--- (1)}$$

If we write $c = a + \theta h$ where $0 < \theta < 1$

Then clearly $c = a + \theta h \in]a, a+h[$.

So (1) becomes

$$f(a+h) - f(a) = h f'(a + \theta h) \quad \text{where } 0 < \theta < 1$$

which is another form of M.V.T.

Cauchy's Mean Value Theorem (OR)

Generalized M.V.T :-

Statement :-

If two function f & ϕ are

- (1) Continuous on $[a, b]$
- (2) Derivable in $]a, b[$
- (3) $\phi'(x) \neq 0$ for all $x \in]a, b[$ Then there

exists at least one pt $c \in]a, b[$ s. that

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(c)}{\phi'(c)}$$

Proof:-

Define a new function

$$F(x) = f(x) + A\phi(x)$$

where A is a const. to be determined
s. that $F(a) = F(b)$

$$\text{So } f(a) + A\phi(a) = f(b) + A\phi(b)$$

$$\text{or } A\phi(b) - A\phi(a) = f(a) - f(b)$$

$$\Rightarrow A(\phi(b) - \phi(a)) = -(f(b) - f(a))$$

Now

$$\phi(b) - \phi(a) \neq 0$$

Because if we suppose that

$$\phi(b) - \phi(a) = 0 \text{ Then } \phi(a) = \phi(b)$$

& then ϕ satisfies all the conditions of Rolle's theorem. So we must have

$$\phi'(c) = 0 \text{ for some } c \in]a, b[$$

which is a contradiction to statement

$$\text{So } \phi(b) - \phi(a) \neq 0$$

$$\Rightarrow A = - \frac{f(b) - f(a)}{\phi(b) - \phi(a)} \quad \text{--- (1)}$$

now the function F being sum of two derivable functions is derivable in $]a, b[$

Hence by Rolle's theorem

$$F'(c) = 0$$

$$\Rightarrow f'(c) + A \phi'(c) = 0$$

$$\Rightarrow A = - \frac{f'(c)}{\phi'(c)} \quad (2)$$

So from (1) & (2)

$$- \frac{f'(c)}{\phi'(c)} = - \frac{f(b) - f(a)}{\phi(b) - \phi(a)}$$

$$\therefore \text{So } \frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(c)}{\phi'(c)} \text{ as desired.}$$

Note :- Another form of Cauchy's M.V.T

If we take the interval as $[a, a+h]$ instead of $[a, b]$. Then Cauchy's M.V.T becomes :-

$$\frac{f(a+h) - f(a)}{\phi(a+h) - \phi(a)} = \frac{f'(c)}{\phi'(c)} \quad (1)$$

where c is a pt s. that $a < c < a+h$

If we write $c = a + \theta h$ where $0 < \theta < 1$

Then clearly $a + \theta h \in]a, a+h[$

So (1) becomes

$$\frac{f(a+h) - f(a)}{\phi(a+h) - \phi(a)} = \frac{f'(a+\theta h)}{\phi'(a+\theta h)} \quad 0 < \theta < 1$$

VII another form of Cauchy's M.V.T.

Increasing & decreasing functions :-

A function is said to be an increasing function on $[a, b]$ if for $x_1, x_2 \in]a, b[$

$$f(x_2) > f(x_1) \text{ whenever } x_2 > x_1$$

& f is said to be a decreasing function on $[a, b]$ if for $x_1, x_2 \in]a, b[$

$$f(x_2) < f(x_1) \text{ whenever } x_2 > x_1$$

Theorem :- Suppose f is continuous on $[a, b]$ & has derivative at each point of $]a, b[$

(1) If f' is +ve in $]a, b[$. Then f is increasing in $[a, b]$.

(2) If f' is -ve in $]a, b[$. Then f is decreasing function in $[a, b]$.

Proof :-

(1) let $x_1, x_2 \in]a, b[$ s. that $x_2 > x_1$
By Lagrange's m.v.T There is pt. c b/w x_1 & x_2 s. that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

$$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \quad \text{--- (1)}$$

Now $f'(c)$ is +ve & also $x_2 - x_1 > 0$

$$\text{So } \textcircled{1} \Rightarrow f(x_2) - f(x_1) > 0$$

$$\text{So } f(x_2) > f(x_1) \quad \text{for } x_2 > x_1$$

Hence f is an increasing function.

② let $x_1, x_2 \in]a, b[$ s. that $x_2 > x_1$

By Lagrange's M.V.T There is a pt c s. that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \quad \text{for } c \in (x_1, x_2)$$

$$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \quad \text{--- } \textcircled{2}$$

$$\text{Now } f'(c) < 0 \quad \& \quad x_2 - x_1 > 0$$

So $f(x_2) < f(x_1)$ whenever $x_2 > x_1$. Thus f is a decreasing function.

Exercise 3.1

- ① Discuss the validity of Rolle's theorem for the following functions.

(i) $f(x) = x^2 - 3x + 2$ on $[1, 2]$

sol

let $f(x) = x^2 - 3x + 2$

$f(1) = 1 - 3 + 2 = 0$

$f(2) = 4 - 6 + 2 = 0$

Thus $f(1) = f(2)$

Clearly $f(x)$ is continuous on $[1, 2]$ & derivable in $]1, 2[$. Since all conditions of Rolle's theorem are satisfied. Hence there must exist a pt. $c \in]1, 2[$ s. that

$f'(c) = 0$

Now $f'(x) = 2x - 3$

$\Rightarrow f'(c) = 2c - 3$

So $2c - 3 = 0$

$\Rightarrow \boxed{c = \frac{3}{2}}$

Hence Rolle's theorem is valid & $c = \frac{3}{2}$

(ii) let $f(x) = \sin^2 x$ on $[0, \pi]$

Now $f(0) = \sin^2 0 = 0$

$f(\pi) = \sin^2 \pi = 0$

$$f(0) = f(\pi)$$

Clearly $f(x)$ is continuous on $[0, \pi]$ and derivable in $]0, \pi[$. Since all conditions of Rolle's theorem are satisfied. Hence there must exist a pt $c \in]0, \pi[$ s. that

$$f'(c) = 0$$

$$\text{now } f'(x) = 2 \sin x \cos x$$

$$\text{so } 2 \sin c \cos c = 0$$

$$\Rightarrow \sin c \cos c = 0$$

$$\Rightarrow \boxed{c = 0, \frac{\pi}{2}}$$

Hence Rolle's theorem is valid at $c = \frac{\pi}{2}$.

$$(iii) \text{ let } f(x) = 1 - x^{3/4} \text{ on } [-1, 1]$$

$$\text{now } f(-1) = 1 - (-1)^{3/4}$$

$$= 1 - (-1)$$

$$f(-1) = 1 + 1 = 2$$

$$f(1) = 1 - (1)^{3/4}$$

$$= 1 - 1$$

$$f(1) = 0$$

$$\text{Hence } f(-1) \neq f(1)$$

Since one of the conditions of Rolle's theorem is not satisfied. Hence Rolle's theorem is not valid & we cannot calculate c .

Sol By Lagrange's Mean Value Theorem

$$f(b) - f(a) = (b-a)f'(c) \quad \text{--- (1)}$$

$$\text{where } c \in]-\frac{11}{7}, \frac{13}{7}[$$

Now

$$f(a) = f(-\frac{11}{7}) = (-\frac{11}{7})^3 - 3(-\frac{11}{7}) - 1$$

$$= -\frac{1331}{343} + \frac{33}{7} - 1$$

$$= \frac{-1331 + 1617 - 343}{343}$$

$$f(-\frac{11}{7}) = -\frac{57}{343}$$

Also

$$f'(x) = 3x^2 - 3$$

$$\Rightarrow f'(c) = 3c^2 - 3$$

putting values in (1)

$$-\frac{57}{343} - (-\frac{57}{343}) = \left[\left(\frac{13}{7}\right) - \left(-\frac{11}{7}\right) \right] (3c^2 - 3)$$

$$-\frac{57}{343} + \frac{57}{343} = \left(\frac{13}{7} + \frac{11}{7} \right) (3c^2 - 3)$$

$$0 = \left(\frac{24}{7} \right) (3c^2 - 3)$$

$$\Rightarrow 3c^2 - 3 = 0$$

$$\Rightarrow c^2 - 1 = 0$$

So

$$\boxed{c = \pm 1}$$

(ii) Here $f(x) = \sin x$ on $[a, b]$

By Lagrange's M.V. Theorem,

$$f(b) - f(a) = (b-a)f'(c) \quad \text{--- (i) for}$$

Some $c \in]a, b[$

Now $f(a) = \sin a$

or $f(b) = \sin b$

also $f'(x) = \cos x$

$$\Rightarrow f'(c) = \cos c$$

So from (i)

$$\sin b - \sin a = (b-a)\cos c \Rightarrow c = \cos^{-1}\left(\frac{\sin b - \sin a}{b-a}\right)$$

(iii) Here $f(x) = 2x - x^3$ on $[0, 1]$

By Lagrange's M.V. Theorem

$$f(b) - f(a) = (b-a)f'(c) \quad \text{--- (i) for}$$

Some $c \in]a, b[$

$$f(a) = f(0) = 0 - 0 = 0$$

$$f(b) = f(1) = 2 - 1 = 1$$

Now

$$f'(x) = 2 - 3x^2$$

$$\Rightarrow f'(c) = 2 - 3c^2$$

Putting values in (i)

$$1 - 0 = (1-0)(2-3c^2)$$

$$\Rightarrow 2 - 3c^2 = 1$$

$$\Rightarrow 3c^2 = 1$$

$$\Rightarrow c^2 = \frac{1}{3}$$

$$c = \pm \frac{1}{\sqrt{3}}$$

$$c = \frac{1}{\sqrt{3}}$$

$$\therefore \frac{1}{\sqrt{3}} \notin]0, 1[$$

(iv) Let $f(x) = x^{2/3}$ on $[-1, 1]$ in

By Lagrange's Mean value Theorem

$$f(b) - f(a) = (b-a) f'(c) \quad \text{--- (1) where } a < c < b$$

now $f(a) = f(-1) = (-1)^{2/3} = 1$

or $f(b) = f(1) = 1$

Now $f(x)$ is not derivable at $x=0$

i.e. $f'(0)$ doesn't exist, where $0 \in]-1, 1[$

Hence mean value theorem is not applicable here.

(3) 'use m.v. Theorem to show' that

(i) $|\sin x - \sin y| \leq |x - y|$ for any real nos x & y

Sol

Let $f(t) = \sin t$

Then $f(t)$ is continuous & differentiable for every real no. Hence Lagrange's m.v. theorem can be applied in the interval $[x, y]$

where x & y are any two real nos.

So $\frac{f(y) - f(x)}{y - x} = \cos z$ where $z \in]x, y[$

but $|\cos z| \leq 1$

$\Rightarrow \left| \frac{\sin y - \sin x}{y - x} \right| \leq 1$

$$\Rightarrow \left| \frac{\sin y - \sin x}{y - x} \right| \leq 1$$

$$\Rightarrow |\sin x - \sin y| \leq |x - y|$$

$$(ii) \left| \frac{\cos ax - \cos bx}{x} \right| \leq |b - a| \text{ if } x \neq 0$$

Sol let $f(t) = \cos(t)$

Then $f(t)$ is continuous & differentiable for every real no. t . Hence Lagrange's M.V.T. can be applied in the interval $[ax, bx]$ where $x \neq 0$ i.e. $\frac{f(bx) - f(ax)}{bx - ax} = -\sin z$ for some $z \in]ax, bx[$

$$\text{or } \left| \frac{\cos bx - \cos ax}{x(b-a)} \right| = |-\sin z|$$

$$\text{or } \left| \frac{\cos bx - \cos ax}{|x| \cdot |b-a|} \right| = |\sin z|$$

$$\text{or } \left| \frac{\cos ax - \cos bx}{x} \right| \cdot \frac{1}{|b-a|} \leq 1 \quad \because |\sin z| \leq 1$$

$$\Rightarrow \left| \frac{\cos ax - \cos bx}{x} \right| \leq (b-a)$$

(iii) $|\tan x + \tan y| \geq |x + y|$, for all $x, y \in \mathbb{R}$ in the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$

Sol :-

$$\text{let } f(t) = \tan t$$

Then $f(t)$ is continuous in $[-x, y]$ & differentiable in $] -x, y [$ where $[-x, y] \subset] -\frac{\pi}{2}, \frac{\pi}{2} [$

Hence Lagrange's m.v.T can be applied in the interval $[-x, y]$.

$$\text{So } \frac{f(y) - f(-x)}{y + x} = \sec^2 z \text{ for some } z \in] -x, y [$$

$$\text{or } \frac{\tan y - \tan(-x)}{y + x} = \sec^2 z$$

$$\text{or } \frac{|\tan y + \tan x|}{|y + x|} = |\sec^2 z|$$

$$\text{But } |\sec^2 z| \geq 1 \quad \forall z \in] -x, y [$$

$$\text{So } \frac{|\tan y + \tan x|}{|y + x|} \geq 1$$

$$\Rightarrow |\tan x + \tan y| \geq |x + y| \quad \forall x, y \in \mathbb{R} \text{ in interval }] -\frac{\pi}{2}, \frac{\pi}{2} [$$

(4) Let a function f be continuous on $[a, b]$ & $f'(x) = 0$ for all $x \in]a, b[$. prove that f is const. use this to show that

$$\sin^2 x + \cos^2 x = 1.$$

Sol :- Given f is continuous on $[a, b]$ & differentiable in $]a, b[$.

Let $x_1, x_2 \in [a, b]$ s. that $x_2 > x_1$

Applying Lagrange's M.V.T on $[x_1, x_2] \subseteq [a, b]$

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \text{ for } c \in]x_1, x_2[$$

$$\text{But } f'(c) = 0$$

$$\Rightarrow f(x_2) - f(x_1) = 0$$

So $f(x_2) = f(x_1)$ whenever $x_2 > x_1$

Hence f is a constt. function.

Now Suppose that $f(x) = \sin^2 x + \cos^2 x$

$$\Rightarrow f'(x) = 2 \sin x \cos x - 2 \cos x \sin x = 0$$

So $f'(x) = 0$ for all real nos. x .

Hence f is constt.

$$\text{Suppose } f(x) = \sin^2 x + \cos^2 x = c \quad \text{--- (1)}$$

where c is an arbitrary constt.

Since eq (1) holds for every real no. x

So for particular $\sin^2 0 + \cos^2 0 = c$

$$\Rightarrow \boxed{c = 1}$$

Hence from (1) $\sin^2 x + \cos^2 x = 1$

③ Show that $f(x) = x^3 - 3x^2 + 3x + 2$ is

monotonically increasing on every interval.

Sol we know that if f is continuous on $[a, b]$ & has derivative at each pt of $]a, b[$. Then f is an increasing function.

if f' is +ve in $]a, b[$

Now $f(x) = x^3 - 3x^2 + 3x + 1$

$$f'(x) = 3x^2 - 6x + 3$$

$$= 3(x^2 - 2x + 1)$$

$$= 3(x-1)^2$$

so $f'(x) = 3(x-1)^2$ is +ve for all real nos x . Hence f is monotonically increasing on every interval.

(Q) prove that $f(x) = 2x - \tan^{-1}x - \ln(x + \sqrt{x^2+1})$ increases steadily on $[0, \infty[$.

Sol Let $f(x) = 2x - \tan^{-1}x - \ln(x + \sqrt{x^2+1})$

Diff w.r. to x

$$f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{x+\sqrt{x^2+1}} \cdot \left[1 + \frac{1}{\sqrt{x^2+1}} \cdot 2x \right]$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{(x+\sqrt{x^2+1})} \cdot \left[\frac{(\sqrt{x^2+1} + x)}{\sqrt{x^2+1}} \right]$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{x^2+1}}$$

$$= \frac{2(1+x^2) - 1 - \sqrt{1+x^2}}{1+x^2}$$

$$\text{or } f'(x) = \frac{2 + 2x^2 - 1 - \sqrt{x^2+1}}{(1+x^2)}$$

Now if $f(x)$ is steadily increasing then $f'(x) \geq 0$

$$\text{i.e. } 2x^2 + 1 - \sqrt{x^2+1} \geq 0$$

$$\text{or } 2(x^2+1) - \sqrt{x^2+1} - 1 \geq 0$$

$$\text{or } 2(x^2+1) - 2\sqrt{x^2+1} + \sqrt{x^2+1} - 1 \geq 0$$

$$\text{or } 2\sqrt{x^2+1}(\sqrt{x^2+1} - 1) + 1(\sqrt{x^2+1} - 1) \geq 0$$

$$\text{or } (2\sqrt{x^2+1} + 1)(\sqrt{x^2+1} - 1) \geq 0$$

which is true for all $x \in [0, \infty[$.

Hence

$f(x)$ increases steadily in $[0, \infty[$

⑦ Show that $\frac{\tan x}{x}$ is an increasing fn for $0 < x < \frac{\pi}{2}$.

Sol :-

$$\text{Let } f(x) = \frac{\tan x}{x}$$

$$\Rightarrow f'(x) = \frac{x \sec^2 x - \tan x}{x^2} \quad \text{--- (1)}$$

$$\text{Let } \phi(x) = x \sec^2 x - \tan x$$

$$\begin{aligned} \phi'(x) &= \sec^2 x + x(2 \sec x \cdot \sec x \tan x) - \sec^2 x \\ &= \sec^2 x + 2x \sec^2 x \tan x - \sec^2 x \end{aligned}$$

$$\text{So } \phi'(x) = 2x \sec^2 x \tan x$$

$$\text{Now clearly } \phi'(x) > 0 \quad \text{when } 0 < x < \frac{\pi}{2}$$

Now

$$\text{obviously } \phi(0) = 0$$

$$\Rightarrow \phi(x) > \phi(0) = 0 \quad \therefore \phi \text{ is increasing fn.}$$

$$\text{i.e. } x \sec^2 x - \tan x > 0$$

$$\Rightarrow x \sec^2 x - \tan x > 0 \quad \text{for } 0 < x < \frac{\pi}{2}$$

Hence from (1) we conclude that

$$f'(x) > 0 \quad \text{for } 0 < x < \frac{\pi}{2}$$

Hence $f(x)$ is an increasing fn for $0 < x < \frac{\pi}{2}$.

⑧ Determine the intervals in which $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing or decreasing.

Sol :-

$$\text{Let } f(x) = 2x^3 - 15x^2 + 36x + 1$$

$$\Rightarrow f'(x) = 6x^2 - 30x + 36$$

If $f(x)$ is an increasing function

$$\text{Then } f'(x) > 0$$

$$\text{i.e. } 6x^2 - 30x + 36 > 0$$

$$\text{or } 6(x^2 - 5x + 6) > 0$$

$$\text{or } (x^2 - 5x + 6) > 0$$

$$\text{or } (x-2)(x-3) > 0$$

$$\text{Now either } x-2 > 0 \text{ \& } x-3 > 0 \left. \vphantom{\begin{array}{l} \text{or} \\ x-2 < 0 \text{ \& } x-3 < 0 \end{array}} \right\}$$

$$\text{or } x-2 < 0 \text{ \& } x-3 < 0$$

$$\text{Case (1) If } x-2 > 0 \text{ \& } x-3 > 0$$

$$\Rightarrow x > 2 \text{ \& } x > 3$$

Hence we see that $f(x)$ is an increasing function for all $x > 3$ i.e. for all $x \in]3, \infty[$

$$\text{Case (2) If } x-2 < 0 \text{ \& } x-3 < 0$$

$$\Rightarrow x < 2 \text{ \& } x < 3$$

So we see that $f(x)$ is an increasing function

for all $x < 2$ i.e. for all $x \in]-\infty, 2[$

Now if $f(x)$ is a decreasing function

$$\text{Then } f'(x) < 0$$

$$\text{i.e. } 6x^2 - 30x + 36 < 0$$

$$\text{or } (x^2 - 5x + 6) < 0$$

$$\text{or } x^2 - 5x + 6 < 0$$

$$\text{or } (x-2)(x-3) < 0$$

$$\text{Now either } (x-2) < 0 \text{ \& } (x-3) > 0 \left. \vphantom{\begin{array}{l} \text{or} \\ (x-2) > 0 \text{ \& } (x-3) < 0 \end{array}} \right\}$$

$$\text{or } (x-2) > 0 \text{ \& } (x-3) < 0$$

Case (1) If $x-2 < 0$ & $x-3 > 0$

$$\Rightarrow x < 2 \text{ & } x > 3$$

which is impossible for any $x \in \mathbb{R}$.

Case (2) If $x-2 > 0$ & $x-3 < 0$

$$\Rightarrow x > 2 \text{ & } x < 3$$

$$\text{So } x \in]2, 3[$$

Hence $f(x)$ is a decreasing function for $x \in]2, 3[$.

(9) If $x > 0$ Then prove that $x - \ln(1+x) > \frac{x^2}{2(1+x)}$

Sol: Let $f(x) = x - \ln(1+x) - \frac{x^2}{2(1+x)}$

$$f'(x) = 1 - \frac{1}{1+x} - \frac{1}{2} \left[\frac{(1+x)2x - x^2 \cdot 1}{(1+x)^2} \right]$$

$$= 1 - \frac{1}{1+x} - \frac{1}{2} \left[\frac{2x + 2x^2 - x^2}{(1+x)^2} \right]$$

$$= 1 - \frac{1}{1+x} - \frac{1}{2} \left(\frac{2x + x^2}{(1+x)^2} \right)$$

$$= 1 - \frac{1}{1+x} - \frac{2x + x^2}{2(1+x)^2}$$

$$= \frac{2(1+x)^2 - 2(1+x) - 2x - x^2}{2(1+x)^2}$$

$$= \frac{2 + 4x + 2x^2 - 2 - 2x - 2x - x^2}{2(1+x)^2}$$

$$f'(x) = \frac{x^2}{2(1+x)^2}$$

obviously $f'(x) > 0$ for all $x > 0$

Hence $f(x)$ is an increasing fn for $x > 0$

But $f(0) = 0$

$\Rightarrow f(x) > f(0)$ for $x > 0$

i.e. $x - \ln(1+x) - \frac{x^2}{2(1+x)} > 0$

i.e. $x - \ln(1+x) > \frac{x^2}{2(1+x)}$ for $x > 0$

Solved Examples :-

(1) verify Rolle's theorem for $f(x) = 1 - x^{2/3}$ on $[-1, 1]$

Sol:- let $f(x) = 1 - x^{2/3}$

$f(-1) = 1 - (-1)^{2/3}$

$= 1 - 1 = 0$

Hence $f(-1) = f(1)$

Clearly $f(x)$ is continuous on $[-1, 1]$ & not derivable in $] -1, 1 [$.

Because $f'(x) = -\frac{2}{3} x^{-1/3}$

$\Rightarrow f'(0) = \frac{-2}{3x^{1/3}} = \frac{2}{0}$

Hence $f(x)$ is not derivable at $x = 0 \in] -1, 1 [$

Rolle's Theorem is not applicable.

(2) If $f(x) = x(x-1)(x-2)$, $a = 0$, $b = \frac{1}{2}$

find c of M.V.T.

Sol:- Here $f(x) = x(x-1)(x-2)$

By Lagrange's M.V.T.

$f(b) - f(a) = (b-a)f'(c)$ ——— (1) for some $c \in] a, b [$

Now

$$f(a) = f(0) = 0$$

$$f(b) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = \frac{3}{8}$$

Also

$$f(x) = x^3 - 3x^2 + 2x$$

$$f'(x) = 3x^2 - 6x + 2$$

$$f'(c) = 3c^2 - 6c + 2$$

put value in (i)

$$\frac{3}{8} - 0 = \left(\frac{1}{2} - 0\right)(3c^2 - 6c + 2)$$

$$\Rightarrow \frac{6}{8} = 3c^2 - 6c + 2$$

$$\text{or } 3c^2 - 6c + 2 = \frac{3}{4}$$

$$12c^2 - 24c + 8 = 3$$

$$12c^2 - 24c + 5 = 0$$

$$c = \frac{24 \pm \sqrt{(24)^2 - 4(12) \cdot 5}}{2 \cdot 12}$$

$$\text{or } c = \frac{24 \pm \sqrt{336}}{24}$$

$$= \frac{24 \pm 4\sqrt{21}}{24}$$

$$c = \frac{6 \pm \sqrt{21}}{6}$$

$$= 1 \pm \frac{\sqrt{21}}{6}$$

$$= 1 \pm \sqrt{\frac{21}{36}}$$

$$c = 1 \pm \sqrt{\frac{7}{12}}$$

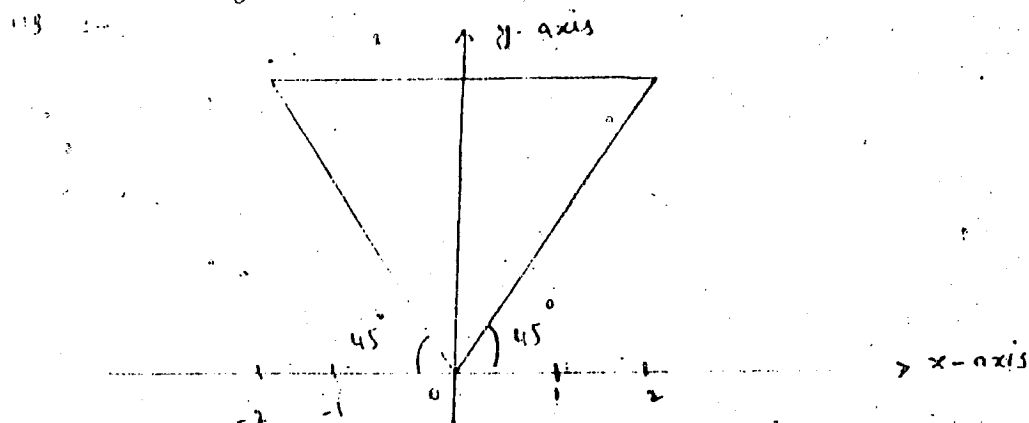
Now $1 + \sqrt{\frac{7}{12}} \notin [0, \frac{1}{2}]$

$\Rightarrow$ $C = 1 - \sqrt{\frac{7}{12}}$ as required.

(3). For the function $f(x) = |x|$. Check whether, M.V.T holds on the interval $[-2, 2]$.

Sol:-

The graph of the function $f(x) = |x|$ is



Now obviously $f(x)$ is continuous on $[-2, 2]$.

The slope of line through pts $A(-2, 2)$ & $B(2, 2)$ is $\frac{2-2}{2-(-2)} = 0$.

Now the function $f(x)$ doesn't have a derivative at $x=0$ because

$$f'(x) \neq 0 \text{ for any } x \in]-2, 2[$$

So one of the conditions of M.V.T is not satisfied. Hence M.V.T does not hold in $[-2, 2]$.

(4) If a function f satisfies the hypothesis of M.V.T on $[a, b]$ & $|f'(x)| \leq M$ for

all $x \in]a, b[$ then prove that

$$|f(b) - f(a)| \leq M(b-a).$$

Sol:- Given that f satisfies the hypothesis of M.V.T on $[a, b]$.

So by M.V.T

$$\frac{f(b) - f(a)}{b-a} = f'(c) \text{ for some } c \in]a, b[.$$

Taking modulus on both sides

$$\left| \frac{f(b) - f(a)}{b-a} \right| = |f'(c)|$$

$$\Rightarrow \left| \frac{f(b) - f(a)}{b-a} \right| \leq M \quad \because |f'(c)| \leq M$$

where $a < c < b$

$$\Rightarrow \frac{|f(b) - f(a)|}{|b-a|} \leq M$$

$$\Rightarrow |f(b) - f(a)| \leq M(b-a) \quad \text{Ans.}$$

⑤ Prove that for $x > 0$ $\frac{x}{x+1} < \ln(x+1) < x$

Sol:- Let $f(x) = \ln(x+1) - \frac{x}{x+1}$

$$f'(x) = \frac{1}{x+1} - \left[\frac{(x+1) \cdot 1 - x}{(x+1)^2} \right]$$

$$= \frac{1}{x+1} - \frac{x+1-x}{(x+1)^2}$$

$$= \frac{1}{x+1} - \frac{1}{(x+1)^2}$$

$$= \frac{x+1-1}{(x+1)^2}$$

$$f'(x) = \frac{x}{(x+1)^2}$$

So $f'(x) > 0$ for all $x > 0$

Hence $f(x)$ is an increasing function.

$$\text{Now } f(0) = 0$$

$$\Rightarrow f(x) > f(0) = 0 \quad \text{for all } x > 0$$

$$\Rightarrow \ln(x+1) - \frac{x}{x+1} > 0$$

$$\Rightarrow \ln(x+1) > \frac{x}{x+1}$$

Hence

$$\frac{x}{x+1} < \ln(x+1) \quad \text{--- (1) for } x > 0$$

$$\text{Again suppose } \phi(x) = x - \ln(x+1)$$

$$\phi'(x) = 1 - \frac{1}{x+1}$$

$$\text{or } = \frac{x+1-1}{x+1} = \frac{x}{x+1}$$

$$\text{so } \phi'(x) > 0 \quad \text{for } x > 0$$

$$\text{Now } \phi(0) = 0$$

$$\text{so } \phi(x) > \phi(0)$$

$$\Rightarrow x - \ln(x+1) > 0$$

$$\text{or } x > \ln(x+1)$$

$$\text{or } \ln(x+1) < x \quad \text{--- (2) if } x > 0$$

Combining (1) & (2), we have

$$\frac{x}{x+1} < \ln(x+1) < x \quad \text{if } x > 0$$

(6) prove that $f(x) = \frac{\ln(x+1)}{x}$ decreases in $]0, \infty[$.

Sol:-

$$\text{Here } f(x) = \frac{\ln(x+1)}{x}$$

Diff. w.r.t. x

$$f'(x) = \frac{x \cdot \frac{1}{x+1} - \ln(x+1)}{x^2}$$

$$= \frac{\frac{x}{x+1} - \ln(x+1)}{x^2}$$

But we know that

$$\frac{x}{x+1} < \ln(x+1) \quad \text{for } x > 0$$

$$\text{so } \frac{x}{x+1} - \ln(x+1) < 0$$

$$\text{Hence } f'(x) = \frac{-ve}{+ve} = -ve \quad \text{for } x > 0$$

Hence $f(x)$ is decreasing function $x > 0$
 i.e., $f(x)$ decreases for $]0, \infty[$.

① Let $f(x) = x^2$ & $\phi(x) = x^3$. Verify Cauchy M.V.T
 in $[1, 2]$. Also find c .

Sol:-

$$\left. \begin{aligned} \text{Given } f(x) &= x^2 \\ \phi(x) &= x^3 \end{aligned} \right\}$$

Obviously $f(x)$ & $\phi(x)$ are continuous in $[1, 2]$ &
 derivable in $]1, 2[$.

Hence by Cauchy's M.V.T

$$\frac{f(2) - f(1)}{\phi(2) - \phi(1)} = \frac{f'(c)}{\phi'(c)} \quad \text{for some } c \in]1, 2[$$

$$\Rightarrow \frac{2^2 - 1^2}{2^3 - 1^3} = \frac{2c}{3c^2}$$

$$\Rightarrow \frac{4-1}{8-1} = \frac{2}{3c}$$

$$\frac{3}{7} = \frac{2}{3c}$$

$$\Rightarrow 9c = 14$$

$$\Rightarrow \boxed{c = \frac{14}{9}}$$

Hence Cauchy's M.V.T holds for $c = \frac{14}{9} \in]1, 2[$