

THE LAPLACE TRANSFORM :-

Chapter 11

Introduction :

Piecewise Continuous:- A real valued function f defined on an interval $[a, b]$ is said to be piecewise continuous in $[a, b]$ if there exists a partition

$P = \{a = x_0, x_1, x_2, \dots, x_{n-1}, x_n = b\}$ of $[a, b]$ such that f is continuous in the interior of each subinterval $[x_i, x_{i+1}]$ and has finite one-sided limits $f(x_i + 0)$ and $f(x_{i+1} - 0)$ at the end points of each subinterval ($i = 0, 1, 2, \dots, n-1$).

The Laplace Transform of f :-

Let f be a real-valued piecewise continuous function defined on $[0, \infty[$. The Laplace transform of f , denoted by $\mathcal{L}(f) = F$, is the function F defined by

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt \quad \text{--- (1)}$$

provided the improper integral in (1) converges.

The domain of F is the set of all real numbers s for which the above integral converges.

Note that the operation transforms the given function f of the variable t into a new function F of the variable s and is written

Symbolically,

$$F(s) = \mathcal{L}\{f(t)\}.$$

Inverse Laplace transform of F :-

If $F = \mathcal{L}\{f\}$ then the original function f is called the inverse Laplace transform of F and is denoted by $f = \mathcal{L}^{-1}\{F\}$. Clearly,

$$\mathcal{L}^{-1}\{\mathcal{L}\{f\}\} = f.$$

Thus if $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Example :-

Let $f(t) = 1$ on $[0, \infty)$. Then

$$\mathcal{L}\{f\} = \mathcal{L}\{1\} = \int_0^{\infty} e^{-st} dt$$

$$= \lim_{h \rightarrow \infty} \int_0^h e^{-st} dt$$

$$= \lim_{h \rightarrow \infty} \left[-\frac{e^{-st}}{s} \right]_0^h$$

$$= \lim_{h \rightarrow \infty} \left[-\frac{e^{-hs}}{s} + \frac{1}{s} \right]$$

$$= \frac{1}{s} = F(s), \text{ provided } s > 0.$$

Example :- Let $f(t) = t^n$, n being a positive integer. Evaluate $\mathcal{L}\{f(t)\}$.

Solution :- Here $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^n\} = \int_0^{\infty} e^{-st} t^n dt$

Integrating by parts, taking t^n as first function

$$\begin{aligned}
 \mathcal{L}\{t^n\} &= \left[t^n \cdot \frac{e^{-st}}{-s} \right]_0^\infty + \int_0^\infty n t^{n-1} \cdot \frac{e^{-st}}{s} dt \\
 &= - \left[\frac{t^n}{s e^{st}} \right]_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt \\
 &= \frac{n}{s} \mathcal{L}\{t^{n-1}\}
 \end{aligned}$$

Hence $\mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\}$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \mathcal{L}\{t^{n-2}\}$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdots \frac{1}{s} \mathcal{L}\{1\}$$

$$= \frac{n!}{s^n} \cdot \frac{1}{s}, \quad (\text{as } \mathcal{L}\{1\} = \frac{1}{s})$$

$$= \frac{n!}{s^{n+1}} = F(s).$$

Example: Compute $\mathcal{L}\{e^{at}\}$, where a is constant and $s \neq a$.

Solution: Here $f(t) = e^{at}$

$$\begin{aligned}
 \mathcal{L}\{f(t)\} &= \mathcal{L}\{e^{at}\} \\
 &= \int_0^\infty e^{-st} \cdot e^{at} dt
 \end{aligned}$$

$$= \int_0^\infty e^{(a-s)t} dt$$

$$= \lim_{h \rightarrow \infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^h$$

$$= \lim_{h \rightarrow \infty} \left[\frac{e^{(a-s)h}}{a-s} - \frac{1}{a-s} \right]$$

$$= \frac{(a-s)t}{a-s} \Big|_0^\infty$$

$$= \frac{1}{a-s} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^\infty$$

$$= \frac{1}{a-s} \left[e^{(a-s)t} \right]_0^\infty$$

$$= \frac{1}{a-s} \left[e^{-\infty} - e^0 \right] = \frac{1}{a-s} [0 - 1] = \frac{-1}{a-s} = \frac{1}{s-a}$$

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$$= \begin{cases} \frac{1}{s-a} & \text{if } s > a \\ 0 & \text{if } s \leq a. \end{cases}$$

Here $e^{-(s-a)h} \rightarrow 0$ as $h \rightarrow \infty$ and $s > a$, while $e^{-(s-a)h} \rightarrow \infty$ as $h \rightarrow \infty$ and $s < a$.

when $s = a$, $f(t) = e^{at}$ and

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} \cdot e^{at} dt = \int_0^{\infty} dt$$

$$= \{t\}_0^{\infty} = \infty.$$

Therefore, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, s > a.$

Example: Find Laplace transforms of (i) $\sin at$, (ii) $\cos at$.

Solution:

By definition,

$$\left. \begin{aligned} \mathcal{L}\{\cos at\} &= \int_0^{\infty} e^{-st} \cos at dt \\ \text{and } \mathcal{L}\{\sin at\} &= \int_0^{\infty} e^{-st} \sin at dt. \end{aligned} \right\}$$

Therefore,

$$\mathcal{L}\{\cos at\} + i \mathcal{L}\{\sin at\} = \int_0^{\infty} e^{-st} \cos at dt + i \int_0^{\infty} e^{-st} \sin at dt$$

$$= \lim_{h \rightarrow \infty} \int_0^h e^{-st} e^{iat} dt = \lim_{h \rightarrow \infty} \int_0^h e^{(ia-s)t} dt$$

$$= \lim_{h \rightarrow \infty} \left[\frac{e^{(ia-s)t}}{ia-s} \right]_0^h = \lim_{h \rightarrow \infty} \left[\frac{e^{(ia-s)h}}{ia-s} - \frac{1}{ia-s} \right]$$

$$= \begin{cases} \frac{1}{s-ia} & \text{if } s > 0 \\ \text{undefined} & \text{if } s < 0 \end{cases}$$

$$= \frac{s+ia}{s^2+a^2} \quad \text{if } s > 0.$$

$$\mathcal{L}\{e^{at} \cos bt\}$$

$$= \int_0^{\infty} e^{-st} \cos bt dt$$

Here $\lim_{h \rightarrow \infty} \frac{(1-a-s)h}{1a-s} = \infty$ for $s > 0$ and is undefined for $s < a$.

Equating real and imaginary parts, we get

$$(i) \quad \mathcal{L}[\cos at] = \frac{s}{s^2 + a^2}, \quad s > 0$$

$$(ii) \quad \mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}, \quad s > 0$$

Example: Consider the function f defined by $f(t) = \frac{1}{t}$. For the Laplace transform of $\frac{1}{t}$, we first check the convergence of $\int_0^{\infty} \frac{e^{-st}}{t} dt$.

$$\int_0^{\infty} \frac{e^{-st}}{t} dt = \int_0^1 \frac{e^{-st}}{t} dt + \int_1^{\infty} \frac{e^{-st}}{t} dt$$

For $0 < t \leq 1$, we have $e^{-st} \geq e^{-s}$ if $s > 0$.

Therefore, $\int_0^{\infty} \frac{e^{-st}}{t} dt \geq \int_0^1 \frac{e^{-s}}{t} dt + \int_1^{\infty} \frac{e^{-st}}{t} dt$

$$\begin{aligned} \text{But } \int_0^1 \frac{e^{-s}}{t} dt &= e^{-s} \lim_{h \rightarrow 0} [\ln t]_h^1 \\ &= \frac{1}{e^s} \lim_{h \rightarrow 0} (\ln 1 - \ln h) \\ &= \infty \end{aligned}$$

Hence $\int_0^{\infty} \frac{e^{-st}}{t} dt$ also diverges to ∞ . Consequently,

$\int_0^{\infty} \frac{e^{-st}}{t} dt$ diverges and so by definition, $\mathcal{L}\left\{\frac{1}{t}\right\}$ does not exist.

Exponential Order a :

A function f defined on $[0, \infty[$ is said to be of exponential order a as $t \rightarrow \infty$ if there exists real constants a , $M > 0$ and $T > 0$ such that

$$|f(t)| \leq M e^{at} \quad \text{for } t \geq T.$$

Theorem: Let f be a piecewise continuous function defined on $[0, \infty[$. If f is of exponential order a as $t \rightarrow \infty$ then $\mathcal{L}\{f(t)\}$ exists for all $s > a$.

Proof: Since f is of exponential order a , there exist positive real numbers M and T such that

$$|f(t)| \leq M e^{at} \quad ; \quad t \geq T \quad \text{--- (1)}$$

The theorem will be proved if we show that $\int_0^\infty e^{-st} f(t) dt$ converges.

$$\text{Now } \int_0^\infty e^{-st} f(t) dt = \int_0^T e^{-st} f(t) dt + \int_T^\infty e^{-st} f(t) dt \quad \text{--- (2)}$$

Since f is piecewise continuous, the first integral on the right of (2) exists. Thus the convergence of $\int_0^\infty e^{-st} f(t) dt$ depends on the convergence of

$$\int_T^\infty e^{-st} f(t) dt \dots \quad \text{But}$$

$$\begin{aligned} \left| \int_T^\infty e^{-st} f(t) dt \right| &\leq \int_T^\infty |e^{-st} f(t)| dt \\ &\leq \int_T^\infty e^{-st} M e^{at} dt, \quad \text{by (1)} \end{aligned}$$

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Let

$$= M \int_0^T e^{(a-s)t} dt$$

$$= M \lim_{h \rightarrow \infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^h$$

$$= \lim_{h \rightarrow \infty} M \left[\frac{e^{(a-s)h}}{a-s} - \frac{e^{(a-s)T}}{a-s} \right]$$

$$= \begin{cases} M \left(0 - \frac{e^{(a-s)T}}{a-s} \right) & \text{if } a-s < 0 \\ \infty & \text{if } a-s \geq 0 \end{cases}$$

Here $\lim_{h \rightarrow \infty} \frac{e^{(a-s)h}}{a-s} = 0$ if $a-s < 0$ and

$\lim_{h \rightarrow \infty} \frac{e^{(a-s)h}}{a-s} = \infty$ if $a-s \geq 0$.

Thus $\mathcal{L}\{f(t)\}$ exists for all $s > a$.

Note: The conditions stated above are sufficient but not necessary. There exists functions which do not satisfy the hypothesis of above theorem.

Example: Consider the function f defined by

$$f(t) = t^{-1/2}$$

clearly f is not defined at $t=0$, but it will be shown that $\mathcal{L}\{t^{-1/2}\}$ exists. By definition, we have

$$\mathcal{L}\{t^{-1/2}\} = \int_0^{\infty} e^{-st} t^{-1/2} dt \quad (1)$$

Let $st = x$. Then $s dt = dx$.

$$\text{or } dt = \frac{1}{s} dx \quad \text{so} \\ t^{-1/2} = \left(\frac{x}{s}\right)^{-1/2} = \sqrt{\frac{s}{x}}$$

Substituting these values of 't' and dt into (1), we have

$$\begin{aligned} \mathcal{L}\{t^{-1/2}\} &= \frac{1}{\sqrt{s}} \int_0^{\infty} e^{-x} x^{-1/2} dx \\ &= \frac{1}{\sqrt{s}} \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{s}} \end{aligned}$$

Thus $\mathcal{L}\{t^{-1/2}\}$ exists.

Properties Of The Laplace Transform:

Theorem: (The linearity property).

Let $f(t) = a g(t) + b h(t)$, where a, b are constants and $\mathcal{L}\{g(t)\}$ and $\mathcal{L}\{h(t)\}$ exists. Then $\mathcal{L}\{f(t)\}$ exists and

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{a g(t) + b h(t)\} \\ &= a \mathcal{L}\{g(t)\} + b \mathcal{L}\{h(t)\} \end{aligned}$$

Proof: By definition,

$$\begin{aligned} \mathcal{L}\{a g(t) + b h(t)\} &= \int_0^{\infty} e^{-st} \{a g(t) + b h(t)\} dt \\ &= a \int_0^{\infty} e^{-st} g(t) dt + b \int_0^{\infty} e^{-st} h(t) dt \\ &= a \mathcal{L}\{g(t)\} + b \mathcal{L}\{h(t)\} \end{aligned}$$

Theorem: (The Differentiation Formula). Let f be continuous on $[0, \infty)$ and of exponential order a . Let f' be piecewise continuous on every finite closed interval $0 \leq t \leq b$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$ and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof: By previous theorem, $\mathcal{L}\{f'(t)\}$ exists.

Let $F(s) = \mathcal{L}\{f(t)\}$. Then by definition,

$$\mathcal{L}\{f'(t)\} = \int_0^{\infty} e^{-st} f'(t) dt$$

$$= \left[e^{-st} f(t) \right]_0^{\infty} + s \int_0^{\infty} e^{-st} f(t) dt$$

$$= -f(0) + s \mathcal{L}\{f(t)\}$$

$$= s \mathcal{L}\{f(t)\} - f(0)$$

$$= sF(s) - f(0).$$

Corollary: If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0).$$

Proof: $\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0),$

$$= s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0)$$

$$= s^2 \mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$= s^2 F(s) - sf(0) - f'(0).$$

Corollary: $\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \dots - f^{(n-1)}(0).$

Proof: $\mathcal{L}\{f^{(n)}(t)\} = s \mathcal{L}\{f^{(n-1)}(t)\} - f^{(n-1)}(0)$
 $= s[s\mathcal{L}\{f^{(n-2)}(t)\} - f^{(n-2)}(0)] - f^{(n-1)}(0)$
 $= s^2 \mathcal{L}\{f^{(n-2)}(t)\} - sf^{(n-2)}(0) - f^{(n-1)}(0)$
 $= s^2[s\mathcal{L}\{f^{(n-3)}(t)\} - f^{(n-3)}(0)] - sf^{(n-2)}(0) - f^{(n-1)}(0)$
 $= s^3 \mathcal{L}\{f^{(n-3)}(t)\} - s^2 f^{(n-3)}(0) - sf^{(n-2)}(0) - f^{(n-1)}(0)$
 Continuing in this way we get the required result.

Theorem (First shifting property) (or Translation property).

Let $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > b$. For any constant a ,

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \text{ for } s > a+b.$$

Proof: By definition,

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^{\infty} e^{-st} e^{at} f(t) dt$$

$$= \int_0^{\infty} e^{-(s-a)t} f(t) dt$$

$$= F(s-a) \text{ if } s-a > b$$

or if $s > a+b$.

Example: Compute $\mathcal{L}\{\sinh at\}$ and $\mathcal{L}\{\cosh at\}$.

Solution: Here $\sinh at = \frac{e^{at} - e^{-at}}{2}$
we have

$$\begin{aligned}\mathcal{L}\{\sinh at\} &= \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\} \\&= \frac{1}{2} \mathcal{L}\{e^{at}\} - \frac{1}{2} \mathcal{L}\{e^{-at}\}, \\&= \frac{1}{2} \cdot \frac{1}{s-a} - \frac{1}{2} \cdot \frac{1}{s+a} \\&= \frac{a}{s^2 - a^2}\end{aligned}$$

Similarly, $\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$

Example: Compute $\mathcal{L}\{\cos at\}$

Solution: Let $f(t) = \cos at$

$$\begin{aligned}f'(t) &= -a \sin at \\&= -a \sin 2at\end{aligned}$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\text{or } s\mathcal{L}\{f(t)\} = \mathcal{L}\{f'(t)\} + f(0)$$

i.e.,

$$s\mathcal{L}\{\cos at\} = -a\mathcal{L}\{\sin 2at\} + f(0)$$

$$= -a \cdot \frac{2a}{s^2 + 4a^2} + 1$$

$$= \frac{s^2 + 2a^2}{s^2 + 4a^2}$$

$$\text{Therefore, } \mathcal{L}\{f(t)\} = \mathcal{L}\{\cos at\}$$

$$= \frac{s^2 + 2a^2}{s(s^2 + 4a^2)}$$

Example: Evaluate, $\mathcal{L}\{e^{3t}(t^3 + \sin 2t)\}$.

Solution:

$$\mathcal{L}\{e^{3t}(t^3 + \sin 2t)\} = \mathcal{L}\{e^{3t}t^3 + e^{3t}\sin 2t\} \\ = \mathcal{L}\{e^{3t}t^3\} + \mathcal{L}\{e^{3t}\sin 2t\}$$

Now $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2 + 4}$$

Now $\mathcal{L}\{e^{3t}t^3\} = \frac{3!}{(s-3)^4}$

and $\mathcal{L}\{e^{3t}\sin 2t\} = \frac{2}{(s-3)^2 + 4}$

Therefore, $\mathcal{L}\{e^{3t}(t^3 + \sin 2t)\} = \frac{3!}{(s-3)^4} + \frac{2}{(s-3)^2 + 4}$

Example 1: Compute $\mathcal{L}\{te^{at}\cos bt\}$.

Solution: Consider $t e^{at} e^{ibt} = t e^{(a+ib)t}$

Let $f(t) = t$, then $\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}$.

Now,

$$\mathcal{L}\{te^{(a+ib)t}\} = \frac{1}{[s - (a+ib)]^2} \quad (\text{by } \mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2})$$

$$= \frac{1}{[(s-a) - ib]^2}$$

$$= \frac{1}{[(s-a) + ib]^2}$$

$$= \frac{1}{[\{(s-a) - ib\} \{(s-a) + ib\}]^2}$$

$$= \frac{(s-a)^2 - b^2 + 2ib(s-a)}{[(s-a)^2 + b^2]^2}$$

Equating real parts, we have

$$\mathcal{L}\{t e^{at} \cos bt\} = \frac{(s-a)^2 - b^2}{[(s-a)^2 + b^2]^2}$$

Theorem: Suppose $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$.
Then $\mathcal{L}\{t f(t)\} = -F'(s)$.

Proof: Consider

$$\begin{aligned} \frac{d}{ds} \mathcal{L}\{f(t)\} &= \frac{d}{ds} F(s) \\ &= \frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^{\infty} \frac{d}{ds} e^{-st} f(t) dt \\ &= \int_0^{\infty} -t e^{-st} f(t) dt \\ &= -\mathcal{L}\{t f(t)\} \end{aligned}$$

$$F'(s) = -\mathcal{L}\{t f(t)\}$$

or $\mathcal{L}\{t f(t)\} = -F'(s)$ as desired.

Using the above result repeatedly, we have

$$\begin{aligned} \mathcal{L}\{t^n f(t)\} &= -\frac{d}{ds} \mathcal{L}\{t^{n-1} f(t)\} \\ &= (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{t^{n-2} f(t)\} \\ &\vdots \\ &= (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}. \end{aligned}$$

Example: Compute $\mathcal{L}\{t^3 e^{-t}\}$.

Solution:

Let $f(t) = e^{-t}$. Then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

$$\begin{aligned}
 \text{Now, } \mathcal{L}\{t^3 e^{-t}\} &= (-1)^3 \cdot \frac{d^3}{ds^3} \mathcal{L}\{e^{-t}\} \\
 &= -\frac{d^3}{ds^3} \left(\frac{1}{s+1} \right) \\
 &= -\frac{(-1)^3 \cdot 3!}{(s+1)^4} \\
 &= \frac{6}{(s+1)^4}
 \end{aligned}$$

Theorem: If $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u) du \quad \text{provided } \lim_{t \rightarrow \infty} \frac{f(t)}{t} \text{ exists}$$

exists.

Proof: Let $\frac{f(t)}{t} = g(t)$. Then $f(t) = tg(t)$.

$$\begin{aligned}
 \text{Now } F(s) &= \mathcal{L}\{f(t)\} = \mathcal{L}\{tg(t)\} \\
 &= -\frac{d}{ds} \mathcal{L}\{g(t)\},
 \end{aligned}$$

Integrating, we have

$$\begin{aligned}
 \mathcal{L}\{g(t)\} &= -\int_s^\infty F(u) du \\
 &= \int_s^\infty F(u) du.
 \end{aligned}$$

Theorem: If f is piecewise continuous and is of exponential order a , then

$$\mathcal{L}\left\{\int_0^t f(u) du\right\} = \frac{1}{s} \mathcal{L}\{f(t)\}.$$

Proof: The integral

$$g(t) = \int_0^t f(u) du$$

is a continuous function of t . Since $f(t)$ is of exponential order a , $|f(t)| \leq M e^{at}$. Therefore,

$$|g(t)| = \left| \int_0^t f(u) du \right| \leq M \int_0^t e^{au} du \\ \leq \frac{M}{a} \{e^{at} - 1\}.$$

By the fundamental theorem of integral calculus, $g'(t) = f(t)$ except at points where f is discontinuous. Hence $g'(t)$ is piecewise continuous. $\therefore$ We have $\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\}$

$$= s \mathcal{L}\{g(t)\} - g(0), \quad s > a$$

$$= s \mathcal{L}\{g(t)\}, \text{ since } g(0) = 0.$$

$$\text{Thus } \mathcal{L}\{g(t)\} = \frac{1}{s} \mathcal{L}\{f(t)\}.$$

Example: Compute $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$.

Solution: Let $f(t) = \sin t$ so that

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1} = F(s).$$

Set $g(t) = \frac{\sin t}{t}$, so we have

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(u) du \\ = \int_s^\infty \frac{1}{1+u^2} du = \left[\arctan u \right]_s^\infty$$

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$$= \frac{\pi}{2} - \operatorname{arctan} s.$$

Example: Evaluate $\mathcal{L} \left\{ \int_0^t \frac{1 - \cosh au}{u} du \right\}$

Solution: Let $f(t) = 1 - \cosh at$ so that

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{s}{s^2 - a^2} = F(s)$$

Set $g(t) = \frac{1 - \cosh at}{t}$. Then, we get

$$\begin{aligned} \mathcal{L} \left\{ \frac{1 - \cosh at}{t} \right\} &= \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2 - a^2} \right) du \\ &= \left[\ln u - \frac{1}{2} \ln(u^2 - a^2) \right]_s^\infty \end{aligned}$$

$$\begin{aligned} &= \lim_{u \rightarrow \infty} \left[\frac{1}{2} \ln u^2 - \frac{1}{2} \ln(u^2 - a^2) \right] \\ &\quad + \frac{1}{2} \ln(s^2 - a^2) - \ln s \end{aligned}$$

$$= \lim_{u \rightarrow \infty} \ln \left(\frac{u^2}{u^2 - a^2} \right) + \frac{1}{2} \ln \left(\frac{s^2 - a^2}{s^2} \right)$$

$$= \ln \lim_{u \rightarrow \infty} \left(\frac{u^2}{u^2 - a^2} \right) + \frac{1}{2} \ln \left(\frac{s^2 - a^2}{s^2} \right)$$

$$= \ln 1 + \frac{1}{2} \ln \left(\frac{s^2 - a^2}{s^2} \right) = \frac{1}{2} \ln \left(\frac{s^2 - a^2}{s^2} \right)$$

$$\begin{aligned} \text{So } \mathcal{L} \left\{ \int_0^t \frac{1 - \cosh au}{u} du \right\} &= \frac{1}{s} \mathcal{L} \left\{ \frac{1 - \cosh at}{t} \right\} \\ &= \frac{1}{2s} \ln \left(\frac{s^2 - a^2}{s^2} \right). \end{aligned}$$

Unit Step Function: Let $a \geq 0$. The function u_a defined on $]0, \infty[$ by

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

is called the unit step function. If $a = 0$, then

$$u_0(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0. \end{cases}$$

Theorem: Let u_a be the unit step function.

Then $\mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s}$.

Proof:

By definition,

$$\begin{aligned} \mathcal{L}\{u_a(t)\} &= \int_0^{\infty} e^{-ts} u_a(t) dt \\ &= \int_0^a e^{-ts} dt + \int_a^{\infty} e^{-ts} dt = \int_a^{\infty} e^{-ts} dt \end{aligned}$$

$$\begin{aligned} &= \lim_{h \rightarrow \infty} \left[\frac{e^{-ts}}{-s} \right]_a^h \\ &= \lim_{h \rightarrow \infty} \left(-\frac{e^{-hs}}{s} + \frac{e^{-as}}{s} \right) \\ &= \frac{e^{-as}}{s}, \quad s > 0, \text{ because } \lim_{h \rightarrow \infty} \frac{e^{-hs}}{s} = 0. \end{aligned}$$

Theorem: Let f be a function of exponential order 'a' and $\mathcal{L}\{f(t)\} = F(s)$. For the function

$$u_a(t)f(t-a) = \begin{cases} 0 & \text{if } 0 < t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

$$\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}F(s).$$

Proof: $\mathcal{L}\{u_a(t)f(t-a)\} = \int_0^\infty e^{-st} u_a(t)f(t-a) dt$

$$= \int_0^a e^{-st} \cdot 0 dt + \int_a^\infty e^{-st} f(t-a) dt$$

$$= \int_a^\infty e^{-st} f(t-a) dt \quad \text{--- (1)}$$

putting $t-a = z$ in (1), we get

$$\begin{aligned} \mathcal{L}\{u_a(t)f(t-a)\} &= e^{-as} \int_0^\infty e^{-sz} f(z) dz \\ &= e^{-as} F(s). \end{aligned}$$

This is known as the second translation property.

Example: Find the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < \frac{\pi}{2} \\ \cos t & \text{if } t > \frac{\pi}{2} \end{cases}$$

Solution: By definition,

$$\begin{aligned} F(s) = \mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt \\ &= \int_0^{\pi/2} e^{-st} f(t) dt + \int_{\pi/2}^\infty e^{-st} f(t) dt \end{aligned}$$

$$\begin{aligned}
 &= 0 + \int_{\pi/2}^{\infty} e^{st} \cos t \, dt \\
 &= \left[\frac{e^{st}}{-s} \cos t \right]_{\pi/2}^{\infty} + \frac{1}{s} \int_{\pi/2}^{\infty} e^{st} (-\sin t) \, dt \\
 &= -\frac{1}{s} \int_{\pi/2}^{\infty} e^{st} \sin t \, dt \\
 &= \frac{1}{s^2} \left[e^{st} \sin t \right]_{\pi/2}^{\infty} - \frac{1}{s^2} \int_{\pi/2}^{\infty} e^{st} \cos t \, dt \\
 &= \frac{e^{-\pi/2 s}}{s^2} - \frac{1}{s^2} F(s)
 \end{aligned}$$

Therefore, $\left(1 + \frac{1}{s^2}\right) F(s) = -\frac{e^{-\pi/2 s}}{s^2}$

or $F(s) = -\frac{e^{-\pi/2 s}}{s^2 + 1}$

Table of Some Laplace Transforms:

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}, s > 0$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}, s > 0$
$t^a, a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$

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 $\mathcal{L}\{f(t)\}$

①

 e^{at}

②

 $t e^{at}$

③

 $\sin at$

④

 $\cos at$

⑤

 $\sinh at$

⑥

 $\cosh at$

⑦

 $t^n e^{at}$

⑧

 $e^{at} \sin bt$

⑨

 $e^{at} \cos bt$

⑩

 $t \sin at$

⑪

 $t \cos at$

⑫

 $f(ct)$

⑬

 $\int_0^t f(u) du$ $\mathcal{L}\{f(t)\} = F(s)$

$$\frac{1}{s-a}, \quad s > a$$

$$\frac{1}{(s-a)^2}$$

$$\frac{a}{s^2+a^2}, \quad s > 0$$

$$\frac{s}{s^2+a^2}, \quad s > 0$$

$$\frac{a}{s^2-a^2}, \quad s > |a|$$

$$\frac{s}{s^2-a^2}, \quad s > |a|$$

$$\frac{n!}{(s-a)^{n+1}}, \quad s > a$$

$$\frac{b}{(s-a)^2+b^2}, \quad s > a$$

$$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$$

$$\frac{2as}{(s^2+a^2)^2}, \quad s > 0$$

$$\frac{s^2-a^2}{(s^2+a^2)^2}, \quad s > 0$$

$$\frac{1}{c} F\left(\frac{s}{c}\right), \quad c > 0$$

$$\frac{1}{s} F(s)$$

19	$t^n f(t)$	$\frac{(-1)^n d^n F(s)}{ds^n}$
20	$\frac{f(t)}{t}$ <i>If limit exists</i> <i>If Laplace exists</i>	$\int_s^\infty F(u) du$
21	$u_a(t)$	$\frac{e^{-as}}{s}$
22	$u_a(t) f(t-a)$	$e^{-as} F(s)$
23	$\sin at - at \cos at$	$\frac{2a^3}{(s^2+a^2)^2}$
24	$f'(t)$	$sF(s) - f(0)$ <i>or</i> $s \mathcal{L}\{f(t)\} - f(0)$
25	$1 - \cos at$	$\frac{a^2}{s(s^2+a^2)}$
26	$at - \sin at$	$\frac{a^3}{s^2(s^2+a^2)}$
27	$\sinh at - \sin at$	$\frac{2a^3}{s^4 - a^4}$
28	$\cosh at - \cos at$	$\frac{2a^2 s}{s^4 - a^4}$