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Linear Diff Eqs

A diff eq of the form $\frac{dy}{dx} + P(x)y = Q(x)$ — (1) (P & Q are fns of x only)
is called a linear diff eq, because it is linear in y and $\frac{dy}{dx}$.

To Solve Multiply both sides of eq (1) by I.F $e^{\int P dx}$ then L.H.S of (1) becomes exact diff of $y + e^{\int P dx}$ i.e. $d(y e^{\int P dx})$ and then Integrating both sides
 $\therefore$ Solution is given by $\int d(y \times \text{I.F.}) = \int Q \times \text{I.F.} dx + C$

Similarly

A diff eq of the form $\frac{dx}{dy} + P(y)x = Q(y)$ — (2) (P & Q are fns of y only)
is called a linear diff eq, because it is linear in x and $\frac{dx}{dy}$.

To solve Multiply both sides of eq (2) by I.F $e^{\int P dy}$, then L.H.S of (2) becomes exact diff of $x + e^{\int P dy}$ i.e. $d(x e^{\int P dy})$ and then Integrating both sides.
 $\therefore$ solution is given by $\int d(x \times \text{I.F.}) = \int Q \times \text{I.F.} dy + C$
E x 9.6

$$\textcircled{1} \frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x} \quad \text{LDE in } y$$

$$\begin{aligned} \text{I.F.} &= e^{\int P dx} = e^{\int \frac{2x+1}{x} dx} = e^{\int (2 + \frac{1}{x}) dx} \\ &= e^{2x + \ln x} = e^{2x} \cdot e^{\ln x} = e^{2x} \cdot x = x e^{2x} \end{aligned}$$

$$\therefore \text{Sol is given by } \int d(y \times \text{I.F.}) = \int Q \times \text{I.F.} dx + C$$

$$\Rightarrow \int d(y e^{2x} x) = \int e^{-2x} \cdot e^{2x} x dx + C$$

$$\Rightarrow y e^{2x} x = \int x dx + C$$

$$\Rightarrow x y e^{2x} = \frac{x^2}{2} + C$$

————— x

$$\textcircled{2} \frac{dy}{dx} + \frac{3}{x} y = 6x^2$$

$$\begin{aligned} \text{I.F.} &= e^{\int P dx} = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = e^{\ln x^3} = x^3 \end{aligned}$$

$$\therefore \text{Sol is given by } \int d(y \times \text{I.F.}) = \int Q \times \text{I.F.} dx + C$$

$$\Rightarrow \int d(y x^3) = \int 6x^2 \cdot x^3 dx + C$$

$$\Rightarrow y x^3 = \int 6x^5 dx + C$$

$$\Rightarrow y x^3 = \frac{6x^6}{6} + C$$

$$\Rightarrow x^3 y = x^6 + C$$

————— x

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$$\textcircled{3} \quad \frac{dy}{dx} + \frac{y}{x \ln x} = \frac{3x^2}{\ln x} \quad (\text{LDE in } y)$$

$$\text{I.F} = e^{\int P dx} = e^{\int \frac{1}{x \ln x} dx} = e^{\frac{\ln x}{x}} = e^{\ln(\ln x)}$$

$$\text{I.F} = e^{\ln(\ln x)} = \boxed{\ln x}$$

$$\text{Sol is given by } \int d(Y \times \text{I.F}) = \int Q \times \text{I.F} dx + C$$

$$\Rightarrow \int d(Y \ln x) = \int \frac{3x^2}{\ln x} \ln x dx + C$$

$$\Rightarrow Y \ln x = \frac{x^3}{3} + C$$

$$Y = \frac{x^3 + C}{\ln x}$$

$$\textcircled{4} \quad \frac{dy}{dx} + 3y = 3x^2 e^{-3x} \quad (\text{LDE in } y)$$

$$\text{I.F} = e^{\int P dx} = e^{\int 3 dx} = \boxed{e^{3x}}$$

$$\text{Sol is given by } \int d(Y \times \text{I.F}) = \int Q \times \text{I.F} dx + C$$

$$\Rightarrow \int d(Y e^{3x}) = \int 3x^2 e^{-3x} e^{3x} dx + C$$

$$Y e^{3x} = \frac{x^3}{3} + C$$

$$Y = \frac{x^3 + C}{e^{3x}}$$

$$\textcircled{7} \quad (x+1) \frac{dy}{dx} - ny = e^x (x+1)^{n+1} \quad (\text{LDE in } y)$$

$$\frac{dy}{dx} - \frac{n}{x+1} y = e^x (x+1)^n$$

$$\text{I.F} = e^{\int P dx} = e^{\int \frac{-n}{x+1} dx} = e^{-n \ln(x+1)} = e^{\ln(x+1)^{-n}} = \frac{1}{(x+1)^n}$$

$$\text{I.F} = (x+1)^{-n} = \boxed{\frac{1}{(x+1)^n}}$$

$$\text{Sol is given by } \int d(Y \times \text{I.F}) = \int Q \times \text{I.F} dx + C$$

$$\Rightarrow \int d\left(Y \cdot \frac{1}{(x+1)^n}\right) = \int e^x (x+1)^n \cdot \frac{1}{(x+1)^n} dx + C$$

$$\frac{Y}{(x+1)^n} = e^x + C$$

$$Y = (e^x + C)(x+1)^n \quad \text{Ans.}$$

$$\textcircled{5} \quad \cos^3 x \frac{dy}{dx} + y \cos x = \sin x$$

$$\frac{dy}{dx} + \frac{y \cos x}{\cos^3 x} = \frac{\sin x}{\cos^3 x}$$

$$\frac{dy}{dx} + \sec^2 x y = \sec^2 x \tan x \quad (\text{LDE in } y)$$

$$\text{I.F} = e^{\int P dx} = e^{\int \sec^2 x dx} = \boxed{e^{\tan x}}$$

$$\text{Sol is given by } \int d(Y \times \text{I.F}) = \int Q \times \text{I.F} dx + C$$

$$\Rightarrow \int d(Y e^{\tan x}) = \int \sec^2 x \tan x e^{\tan x} dx + C$$

$$\Rightarrow Y e^{\tan x} = \int e^t \frac{t}{\sec^2 x} dt + C \quad \begin{matrix} \tan x = t \\ \sec^2 x dx = dt \end{matrix}$$

$$= t e^t - \int 1 \cdot e^t dt + C$$

$$= t e^t - e^t + C$$

$$Y e^{\tan x} = e^t (t - 1) + C$$

$$Y e^{\tan x} = e^{\tan x} (\tan x - 1) + C$$

$$Y = (\tan x - 1) + C e^{-\tan x}$$

$$\textcircled{6} \quad x \frac{dy}{dx} + (1 + x \cot x) y = x \quad (\text{LDE in } y)$$

$$\frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right) y = 1$$

$$\text{I.F} = e^{\int P dx} = e^{\int \left(\frac{1}{x} + \cot x\right) dx} = e^{\ln x + \ln \sin x} = e^{\ln(x \sin x)} = \boxed{x \sin x}$$

$$\text{Sol is given by } \int d(Y \times \text{I.F}) = \int Q \times \text{I.F} dx + C$$

$$\Rightarrow \int d(Y x \sin x) = \int x \sin x dx + C$$

$$Y x \sin x = x(-\cos x) - \int 1 \cdot (-\cos x) dx$$

$$= x(-\cos x) + \int \cos x dx$$

$$Y x \sin x = -x \cos x + \sin x + C$$

$$Y = -\cot x + \frac{1}{x} + \frac{C}{x \sin x}$$

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$$(8) (x^2+1) \frac{dy}{dx} + 2xy = 4x^2$$

$$\frac{dy}{dx} + \left(\frac{2x}{x^2+1}\right)y = \frac{4x^2}{x^2+1} \quad (\text{LDE in } y)$$

$$\text{I.F} = e^{\int \left(\frac{2x}{x^2+1}\right) dx} = e^{\ln(x^2+1)} = \boxed{x^2+1}$$

$$\text{Sol is given by } \int d(Y \cdot \text{I.F}) = \int Q \cdot \text{I.F} dx + C$$

$$\Rightarrow \int d(Y(x^2+1)) = \int \frac{4x^2}{x^2+1} (x^2+1) dx + C$$

$$Y(x^2+1) = \frac{4x^3}{3} + C$$

$$3Y(x^2+1) = 4x^3 + C'$$

$$(1) x \frac{dy}{dx} + 2y = \sin x$$

$$\frac{dy}{dx} + \frac{2}{x} y = \frac{\sin x}{x} \quad (\text{LDE in } y)$$

$$\text{IF} = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = \boxed{x^2}$$

$$\text{Sol is given by } \int d(Y \cdot \text{I.F}) = \int Q \cdot \text{I.F} dx + C$$

$$\Rightarrow \int d(Y x^2) = \int \frac{\sin x}{x} x^2 dx + C$$

$$Y x^2 = \int x \sin x dx + C$$

$$Y x^2 = x(-\cos x) - \int 1(-\cos x) dx + C$$

$$Y = \frac{1}{x^2} (-x \cos x + \sin x + C)$$

$$(1c) (1+x^2) \frac{dy}{dx} + 4xy = \frac{1}{(1+x^2)^2}$$

$$\frac{dy}{dx} + \left(\frac{4x}{1+x^2}\right)y = \frac{1}{(1+x^2)^3} \quad (\text{LDE in } y)$$

$$\text{I.F} = e^{\int \frac{4x}{1+x^2} dx} = e^{2 \ln(1+x^2)} = e^{\ln(1+x^2)^2} = \boxed{(1+x^2)^2}$$

$$\text{Sol is given by } \int d(Y \cdot \text{I.F}) = \int Q \cdot \text{I.F} dx + C$$

$$\Rightarrow \int d(Y(1+x^2)^2) = \int \frac{1}{(1+x^2)^3} (1+x^2)^2 dx + C$$

$$Y(1+x^2)^2 = \int \frac{dx}{1+x^2} + C \Rightarrow Y = \frac{1}{(1+x^2)} \left[\tan^{-1} x + C \right] \text{ Ans.}$$

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$$\frac{dy}{dx} = \frac{1}{e^y - x}$$

$$\frac{dx}{dy} = e^y - x$$

Reciprocal

$$\frac{dx}{dy} + x = e^y \quad (\text{LDE in } x)$$

$$\text{IF} = e^{\int 1 \cdot dy} = \boxed{e^y}$$

$$\text{Sol is given by } \int d(x \cdot \text{I.F}) = \int Q \cdot \text{I.F} dy + C$$

$$\Rightarrow \int d(x e^y) = \int e^y e^y dy + C$$

$$\Rightarrow x e^y = \int e^{2y} dy + C$$

$$x = \frac{1}{e^y} \left(\frac{e^{2y}}{2} + C \right)$$

$$x = \frac{e^y}{2} + C e^{-y}$$

$$(12) (x+2y^3) \frac{dy}{dx} = y$$

$$\left(\frac{1}{x+2y^3}\right) \frac{dx}{dy} = \frac{1}{y}$$

$$\frac{dx}{dy} = \frac{x+2y^3}{y}$$

$$\frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \left(\frac{1}{y}\right)x = 2y^2 \quad (\text{LDE in } x)$$

$$\text{I.F} = e^{\int \left(-\frac{1}{y}\right) dy} = e^{-\ln y} = e^{\ln y^{-1}} = \frac{1}{y} = \boxed{\frac{1}{y}}$$

$$\text{Sol is given by } \int d(x \cdot \text{I.F}) = \int Q \cdot \text{I.F} dy + C$$

$$\Rightarrow \int d\left(x \cdot \frac{1}{y}\right) = \int 2y^2 \cdot \frac{1}{y} dy + C$$

$$\Rightarrow \frac{x}{y} = \int 2y dy + C$$

$$\Rightarrow x = y \left(\frac{2y^2}{2} + C \right)$$

$$\Rightarrow x = y^3 + Cy$$

Bernoulli Eq

is the diff eq of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad \text{--- ①}$$

To Solve

① Divide the eq ① by y^n

$$\Rightarrow y^{-n} \frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

② Multiply both sides by $(1-n)$

$$\Rightarrow (1-n)y^{-n} \frac{dy}{dx} + P(x)y^{1-n}(1-n) = (1-n)Q(x)$$

③ Put $y^{1-n} = v$

$$\text{+ diff } (1-n)y^{-n} \frac{dy}{dx} = \frac{dv}{dx}$$

$\Rightarrow$

$$\frac{dv}{dx} + P(x)v(1-n) = (1-n)Q(x)$$

which is L.D.E in v

Now solve it easily as before.

Note

The diff eq of the form
is also called Bernoulli Eq

$$\frac{dx}{dy} + P_1 y x = Q_1 y x^n$$

$$\text{⑬ } x \frac{dy}{dx} + y = y^2 \ln x$$

$$\div \text{ by } x \quad \frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{\ln x}{x} y^2 \quad \text{Bernoulli Eq.}$$

$\div \text{ by } y^2$

$$y^{-2} \frac{dy}{dx} + \left(\frac{1}{x}\right)y^{-1} = \frac{\ln x}{x}$$

$$\times \text{ by } -1 \quad -y^{-2} \frac{dy}{dx} - \frac{1}{x} y^{-1} = -\frac{\ln x}{x}$$

Put $y^{-1} = v$

$$-y^{-2} \frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{dv}{dx} + \left(-\frac{1}{x}\right)v = -\frac{\ln x}{x} \quad (\text{LDE in } v)$$

$$\text{I.F} = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = \boxed{\frac{1}{x}}$$

$$\text{Solving by } \int d(v \cdot \frac{1}{x}) = \int -\frac{\ln x}{x} \cdot \frac{1}{x} dx + C$$

$$\Rightarrow \frac{v}{x} = -\int \ln x \cdot \frac{1}{x^2} dx + C$$

$$\frac{v}{x} = -\left(\ln x \cdot \frac{x^{-1}}{-1} - \int \frac{1}{x} \cdot \frac{x^{-1}}{-1} dx\right) + C$$

$$\frac{y^{-1}}{x} = \frac{1}{x} \ln x - \int x^{-2} dx + C$$

$$\frac{1}{xy} = \frac{1}{x} \ln x - \frac{x^{-1}}{-1} + C$$

$$\frac{1}{y} = \frac{x \ln x}{x} + x x^{-1} + C x$$

$$\frac{1}{y} = \ln x + 1 + C x$$



1) $\frac{dy}{dx} + y = xy^3$ Bernoulli Eq

by $y^{-3} \frac{dy}{dx} + y^{-2} = x$

by $(-2) \frac{dy}{dx} + (-2)y^{-2} = -2x$

let $y^{-2} = v$

$-2y^{-3} \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} + (-2)v = -2x$ (LDE in v)

I.F = $e^{\int -2dx} = e^{-2x}$

sol is given by $\int d(v e^{-2x}) = \int (-2x) e^{-2x} dx + C$

$\Rightarrow v e^{-2x} = \int t e^{\frac{t}{-2}} \frac{dt}{-2} + C$

Put $-2x = t$
 $-2dx = dt$
 $dx = \frac{dt}{-2}$

$\Rightarrow v e^{-2x} = -\frac{1}{2} \int t e^{\frac{t}{-2}} dt + C$

$\Rightarrow v e^{-2x} = -\frac{1}{2} \left[t e^{\frac{t}{-2}} - \int 1 \cdot e^{\frac{t}{-2}} dt \right] + C$

$\Rightarrow v e^{-2x} = -\frac{1}{2} \left[t e^{\frac{t}{-2}} - e^{\frac{t}{-2}} \right] + C$

$\Rightarrow v e^{-2x} = -\frac{1}{2} e^{\frac{t}{-2}} (t - 1) + C$

$\Rightarrow \frac{1}{y^2} e^{-2x} = -\frac{1}{2} e^{-2x} (-2x - 1) + C$

$\Rightarrow \frac{1}{y^2} = -\frac{1}{2} \frac{e^{-2x}}{e^{-2x}} (-2x - 1) + \frac{C}{e^{-2x}}$

$\Rightarrow \frac{1}{y^2} = -\frac{1}{2} (-2x - 1) + C e^{2x}$

15) $x \frac{dy}{dx} - 2x^2 y = y \ln y$

$\frac{dy}{dx} - 2xy = \frac{y \ln y}{x}$

$\frac{1}{y} \frac{dy}{dx} - 2x = \frac{\ln y}{x}$

Put $\ln y = v$

$\frac{1}{y} \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} - 2x = \frac{v}{x}$

$\frac{dv}{dx} - \frac{v}{x} = 2x$ (LDE in v)

I.F = $e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$

sol is given by $\int d(v x^{-1}) = \int 2x x^{-1} dx + C$

$\Rightarrow v x^{-1} = \int 2 dx + C$

$\Rightarrow \frac{v}{x} = 2x + C$

$\Rightarrow \frac{\ln y}{x} = 2x + C$

$\Rightarrow \ln y = 2x^2 + Cx$

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$$(16) (x^2+1) \frac{dy}{dx} + 4xy = x, \quad y(2)=1$$

$$\frac{dy}{dx} + \left[\frac{4x}{x^2+1} \right] y = \frac{x}{(x^2+1)} \quad (\text{LDE in } y)$$

$$\text{I.F.} = e^{\int \frac{4x}{x^2+1} dx} = e^{2 \ln(x^2+1)} = e^{\ln(x^2+1)^2} = (x^2+1)^2$$

$$\text{Solving by } \int d(y(x^2+1)^2) = \int \frac{x}{(x^2+1)} dx + C$$

$$\Rightarrow y(x^2+1)^2 = \int x(x^2+1) dx + C$$

$$\Rightarrow y(x^2+1)^2 = \int (x^3+x) dx + C$$

$$\Rightarrow y(x^2+1)^2 = \frac{x^4}{4} + \frac{x^2}{2} + C$$

$$\because y(2)=1$$

$$\therefore 1(25) = 6 + C$$

$$\boxed{19 = C}$$

$$\therefore y(x^2+1)^2 = \frac{x^4}{4} + \frac{x^2}{2} + 19$$

$$(19) x(2+x) \frac{dy}{dx} + 2(1+x)y = 1+3x^2, \quad y(-1)=1$$

$$\frac{dy}{dx} + \left[\frac{2(1+x)}{x(2+x)} \right] y = \frac{1+3x^2}{x(2+x)} \quad (\text{LDE in } y)$$

$$\text{I.F.} = e^{\int \frac{2+2x}{2x+x^2} dx} = e^{\ln(2x+x^2)} = (2x+x^2)$$

$$\text{Solving by } \int d(y(2x+x^2)) = \int \frac{1+3x^2}{x(2+x)} (2x+x^2) dx + C$$

$$\Rightarrow y(2x+x^2) = x + \frac{x^3}{3} + C$$

$$\Rightarrow y = \frac{x+x^3}{2x+x^2}$$

$$\because y(-1)=1$$

$$\therefore 1 = \frac{-1-1+C}{-2+1}$$

$$-1 = -2 + C$$

$$\boxed{1 = C}$$

$$\therefore y = \frac{x+x^3}{2x+x^2} \text{ Ans.}$$

$$(17) e^x (y - 3(e^x+1)^2) dx + (e^x+1) dy = 0, \quad y(0)=4$$

$$(e^x+1) dy = -e^x (y - 3(e^x+1)^2) dx$$

$$\frac{dy}{dx} = -\frac{e^x}{e^x+1} \left[y - 3(e^x+1)^2 \right]$$

$$\frac{dy}{dx} = -\left(\frac{e^x}{e^x+1} \right) y + \frac{3e^x(e^x+1)^2}{(e^x+1)}$$

$$\frac{dy}{dx} + \left(\frac{e^x}{e^x+1} \right) y = 3e^x(e^x+1) \quad (\text{LDE in } y)$$

$$\text{I.F.} = e^{\int \frac{e^x}{e^x+1} dx} = e^{\ln(e^x+1)} = (e^x+1)$$

$$\text{Solving by } \int d(y(e^x+1)) = \int 3e^x(e^x+1)^2 dx + C$$

$$\Rightarrow y(e^x+1) = \frac{3(e^x+1)^3}{3} + C$$

$$\Rightarrow y = (e^x+1)^2 + C$$

$$\because y(0)=4$$

$$4 = (1+1)^2 + C$$

$$\boxed{0 = C}$$

$$\Rightarrow y = (e^x+1)^2 \text{ Ans.}$$

(18) $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}$, $y(1) = 2$ (58)

$\frac{dy}{dx} + \frac{1}{2x} y = x y^{-3}$ Bernoulli Eq.

$\div \text{by } y^{-3}$ $3 y \frac{dy}{dx} + \frac{1}{2x} y^4 = x$

$\times \text{by } 4$ $4 y^3 \frac{dy}{dx} + \frac{2}{x} y^4 = 4x$

Put $y^4 = v$

$4 y^3 \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} + \frac{2}{x} v = 4x$ (LDE in V)

I.F = $e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2$

Sol given by $\int d(v x^2) = \int (4x) x^2 dx + C$

$\Rightarrow v x^2 = \int 4x^3 dx + C$

$y^4 x^2 = \frac{4x^4}{4} + C$

$\because y(1) = 2$

$(2)^4 \cdot 1^2 = 1^4 + C$

$16 - 1 = C$

$15 = C$

$y^4 x^2 = x^4 + 15$

Bernoulli Eq

(20) $\frac{dy}{dx} + \frac{3y}{x} = x^2 y^2$, $y(1) = 2$

$\div \text{by } y^2$ $y^2 \frac{dy}{dx} + \frac{3}{x} y^{-1} = x^2$

$\times \text{by } (-1)$ $-\frac{1}{y} \frac{dy}{dx} - \frac{3}{x} y^{-1} = -x^2$

Put $y^{-1} = v$

$-\frac{1}{y^2} \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} - \frac{3}{x} v = -x^2$ (LDE in V)

I.F = $e^{\int -\frac{3}{x} dx} = e^{-3 \ln x} = e^{\ln x^{-3}} = x^{-3}$

Sol given by $\int d(v x^{-3}) = \int (-x^2) x^{-3} dx + C$

$\Rightarrow v x^{-3} = -\int x^{-1} dx + C$

$\Rightarrow \frac{v}{x^3} = -\int \frac{dx}{x} + C$

$\Rightarrow \frac{1}{y x^3} = -\ln x + C$

$\Rightarrow \frac{1}{y} = x^3 (\ln x^{-1} + C)$

$\because y(1) = 2$

$\therefore \frac{1}{2} = 1 (\ln 1^{-1} + C)$

$\frac{1}{2} = C$

$\therefore \frac{1}{y} = x^3 (\ln x^{-1} + \frac{1}{2})$