

Exercise 6.7

Parametric representation of curves

Let $y = f(x)$ ——— I

be any curve in rectangular coordinates.

Then $x = x(t)$ and $y = y(t)$

represent parametric form of (I)

We know that Eq. of the circle is

$$x^2 + y^2 = a^2$$

Parametric form of this circle is

$$x = a \cos t, \quad y = a \sin t.$$

Similarly parametric form the parabola

$$y^2 = 4ax$$

is

$$x = at^2, \quad y = 2at.$$

Similarly parametric form of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is

$$x = a \cos \theta, \quad y = b \sin \theta.$$

The Cycloid.

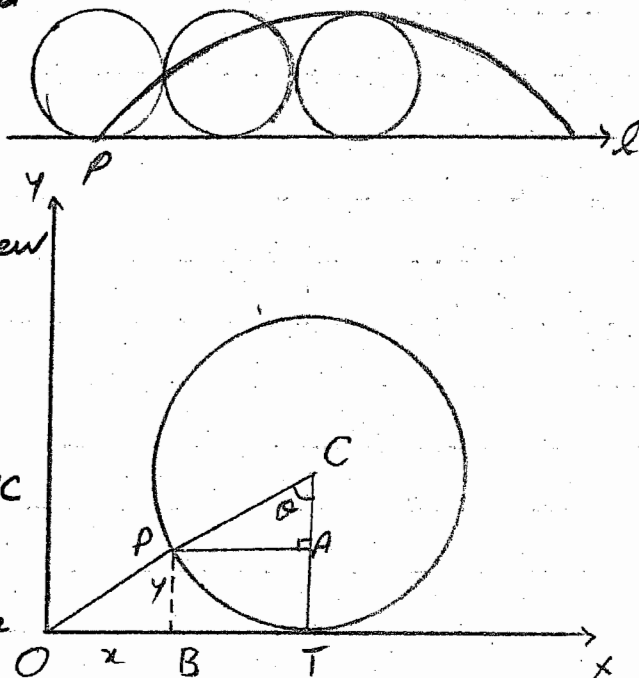
Def: The cycloid is a curve described by a point marked on the circumference of a circle as it rolls without slipping or sliding along a fixed right line.

Let the line l be taken as x -axis and let the marked point P be at O . Let the new position be as shown.

Draw PA and $PB \perp$ to CT and x -axis.

Let $\angle PCA = \theta$ and $CT = a = PC$ the radius of the circle. Let the coordinates of P are (x, y) distance

We note that the distance travelled along x -axis = OT



and Distance along the arc of the circle = PT

$$OT = \widehat{PT} = a\theta$$

$$OT = a\theta$$

$$l = r\theta$$

$$x = OB = OT - BT$$

$$x = OT - PA$$

$$x = a\theta - a \sin\theta$$

$$\therefore PT = BT$$

$$\frac{PA}{PC} = \sin\theta$$

$$PA = \sin\theta \cdot PC$$

Now

$$y = PB = AT$$

$$y = CT - CA$$

$$y = a - a \cos\theta$$

$$\text{So } x = a(\theta - \sin\theta)$$

$$y = a(1 - \cos\theta)$$

are the parametric equations of the cycloid.

Exercise 6.7

Exercise 6.7.

Find parametric eqs. of the given curves (1-3)

① #1

Soln.

For parametric eqs.

Put $x = r \cos \theta$

$$\Rightarrow x = a \sin 2\theta \cos \theta$$

and $y = r \sin \theta$

$$\Rightarrow y = a \sin 2\theta \sin \theta$$

Hence $x = a \sin 2\theta \cos \theta$

$$y = a \sin 2\theta \sin \theta \text{ are the required}$$

parametric eqs.

② #2

Soln.

$$r = \theta$$

$$x = r \cos \theta$$

$$x = \theta \cos \theta$$

and $y = r \sin \theta$

$$y = \theta \sin \theta$$

So parametric eqs. of $r = \theta$ are

$$x = \theta \cos \theta$$

$$y = \theta \sin \theta$$

③ #3

Soln.

$\therefore$ For parametric eqs.

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\Rightarrow x = (2 + 3 \sin \theta) \cos \theta, \quad y = (2 + 3 \sin \theta) \sin \theta$$

are the required parametric eqs.

④ #4

Show that the equations

$$x = a + r \cos \theta, \quad y = b + r \sin \theta$$

are parametric eqs. for a circle with centre (a, b) and radius $|r|$.

Soln.

Here $x = a + r \cos \theta$ _____ (1)

$$y = b + r \sin \theta$$
 _____ (2)

$$1) \Rightarrow x - a = r \cos \theta$$
 _____ (3)

$$2) \Rightarrow y - b = r \sin \theta$$
 _____ (4)

Now $(3)^2 + (4)^2 \Rightarrow$

$$(x-a)^2 + (y-b)^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$(x-a)^2 + (y-b)^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$(x-a)^2 + (y-b)^2 = r^2$$

which represent a circle with centre at (a, b) and radius $|r|$.

③ #5 Show that the curve whose parametric eqs. are

$$\left. \begin{aligned} x &= a \cos \theta + h \\ y &= b \sin \theta + k \end{aligned} \right\}, \quad 0 \leq \theta \leq 2\pi$$

is an ellipse with centre (h, k)

Soln:

Here $x = a \cos \theta + h$ _____ (1)

$$y = b \sin \theta + k \quad \text{_____ (2)}$$

$$1) \Rightarrow x - h = a \cos \theta$$

$$\frac{x-h}{a} = \cos \theta \quad \text{_____ (3)}$$

$$2) \Rightarrow y - k = b \sin \theta$$

$$\frac{y-k}{b} = \sin \theta \quad \text{_____ (4)}$$

Now $(3)^2 + (4)^2 \Rightarrow$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \cos^2 \theta + \sin^2 \theta$$

$$\Rightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

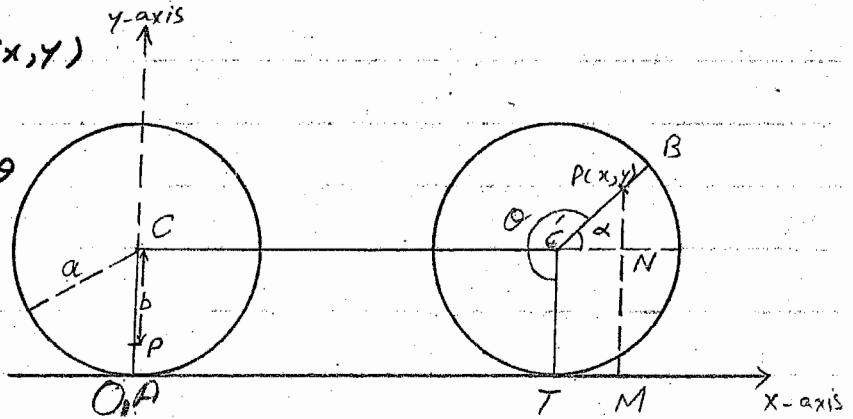
which represent an ellipse with centre (h, k) .

④ #6. A wheel of radius a rolls on a straight line without slipping or sliding. Let P be a fixed point on the wheel, at a distance b from the centre of the wheel. Find parametric eqs. of the curve described by the point P . The curve is called a trochoid. Hence deduce parametric equations for a cycloid.

Soln: Let the wheel be initially in contact with the line at O . where the line is x -axis, $CP = b$. Let the wheel rotate through an angle θ as reflected in the end fig.

Let $\widehat{PCN} = \alpha$ and $P = (x, y)$
 then $\widehat{BCT} = \theta$

and $OT = TB = a\theta \quad \therefore l = r\theta$
 (because the radial distance of P is equal to the distance along x-axis)



Now $x = OT + TM$

$x = OT + C'N$

$x = a\theta + b \cos \alpha \quad \text{--- (1)}$

and $y = PN + NM$

$\therefore NM = C'T = a$

$y = b \sin \alpha + a \quad \text{--- (2)}$

$\therefore \alpha + \theta = \frac{3\pi}{2}$

$\Rightarrow \alpha = \frac{3\pi}{2} - \theta$

Putting $\alpha = \frac{3\pi}{2} - \theta$ into (1) and (2), we have

$x = a\theta + b \cos(\frac{3\pi}{2} - \theta) \Rightarrow x = a\theta - b \sin \theta$

$y = b \sin(\frac{3\pi}{2} - \theta) + a \Rightarrow y = -b \cos \theta + a$

Hence the parametric eqs. of the cycloid are

$x = a(\theta) - b \sin \theta \quad \text{--- (3)}$

$y = a - b \cos \theta \quad \text{--- (4)}$

Put $b = a$ in (3) and (4), we get

$x = a\theta - a \sin \theta \Rightarrow x = a(\theta - \sin \theta)$

$y = a - a \cos \theta \Rightarrow y = a(1 - \cos \theta)$

So $x = a(\theta - \sin \theta)$

$y = a(1 - \cos \theta)$

are eqs. of a cycloid.

Q#7. Find the points at which $r = 1 + \cos \theta$ has horizontal and vertical tangents.

Here $r = 1 + \cos \theta$ — (1)

Diff. w.r.t. ' θ ' $\frac{dr}{d\theta} = -\sin \theta$

$\Rightarrow \frac{d\theta}{dr} = -\frac{1}{\sin \theta}$

So $\tan \psi = r \frac{d\theta}{dr}$
 $= 1 + \cos \theta \cdot -\frac{1}{\sin \theta}$

$= -\frac{2 \cos^2 \theta/2}{2 \sin \theta/2 \cos \theta/2}$

$= -\cot \theta/2$

$= \tan(\frac{\pi}{2} + \theta/2)$

$\Rightarrow \psi = \frac{\pi}{2} + \theta/2$

But $\alpha = \theta + \psi$

$\Rightarrow \alpha = \theta + \frac{\pi}{2} + \theta/2$

$\alpha = \frac{3\theta}{2} + \frac{\pi}{2}$ — (2)

For horizontal tangents.

For these tangents we may have the following two cases.

Case 1. If $\alpha = 0$ Put in (2)

$2) \Rightarrow 0 = \frac{3\theta}{2} + \frac{\pi}{2}$

$0 = 3\theta + \pi$

$\theta = -\frac{\pi}{3} = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\Rightarrow$ at $\theta = -\frac{\pi}{3}$ the tangent is $\parallel$ to x-axis.

Put $\frac{5\pi}{3}$ in (1)

$\Rightarrow r = \frac{3}{2}$

$\Rightarrow$ the tangent is horizontal at $(\frac{3}{2}, \frac{5\pi}{3})$

Case 2. If $\alpha = \pi$

Put in (2) $\Rightarrow \pi = \frac{3\theta}{2} + \frac{\pi}{2}$

$\frac{3\theta}{2} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

$\Rightarrow$ at $\theta = \frac{\pi}{3}$ the tangent is horizontal.

Put $\frac{\pi}{3}$ in (1)

$\Rightarrow r = \frac{3}{2}$

$\Rightarrow$ the tangent is horizontal at $(\frac{3}{2}, \frac{\pi}{3})$

For Vertical Tangents

Case 1

Two cases arise here

if $\alpha = \frac{\pi}{2}$, put in (2)

$$\Rightarrow \frac{\pi}{2} = \frac{3\theta}{2} + \frac{\pi}{2}$$

$$\Rightarrow \frac{3\theta}{2} = 0$$

$$\Rightarrow \theta = 0$$

$\Rightarrow$ at $\theta = 0$ the tangent is vertical

Case 2

if $\alpha = \frac{3\pi}{2}$, put in (2)

$$\Rightarrow \frac{3\pi}{2} = \frac{3\theta}{2} + \frac{\pi}{2}$$

$$\Rightarrow \frac{3\theta}{2} = \frac{3\pi}{2} - \frac{\pi}{2}$$

$$\Rightarrow 3\theta = 3\pi - \pi$$

$$\Rightarrow 3\theta = 2\pi$$

$$\Rightarrow \theta = \frac{2\pi}{3}$$

$\Rightarrow$ at $\theta = \frac{2\pi}{3}$ the tangent is vertical.

Q#8 Find the points on the curve

$$x(t) = t^2 + 4, \quad y(t) = 3t^2 - 6t + 2$$

where tangents are horizontal and vertical.

Soln:

Here $x = t^2 + 4$ ——— (i)

and $y = 3t^2 - 6t + 2$ ——— (ii)

Now Diff. (i) w.r.t. t

$$\frac{dx}{dt} = 2t \quad \text{——— (iii)}$$

Also Diff. (ii) w.r.t. t

$$\frac{dy}{dt} = 6t - 6 \quad \text{——— (iv)}$$

$$\text{Now } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t-6}{2t} = \frac{6(t-1)}{2t}$$

$$\frac{dy}{dx} = \frac{3(t-1)}{t} \quad \text{——— (v)}$$

Now for horizontal tangent

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dt} = 0 \quad \text{and} \quad \frac{dx}{dt} \neq 0$$

Put $\theta = 0$ in (1)

$$\Rightarrow r = 2$$

$\Rightarrow$ the tangent is vertical at $(2, 0)$

Put $\theta = \frac{2\pi}{3}$ in (1)

$$\Rightarrow r = \frac{1}{2}$$

$\Rightarrow$ the tangent is vertical at $(\frac{1}{2}, \frac{2\pi}{3})$

So for horizontal tangent-

$$\frac{dy}{dt} = 0$$

$$6t - 6 = 0$$

$$6t = 6$$

$$\Rightarrow t = 1$$

Put $t = 1$ in (1) and (2)

$$\Rightarrow x = 1 + 4 = 5$$

$$\text{and } y = 3 - 6 + 2 = -1$$

$\Rightarrow$ at $(5, -1)$ the tangent is $\parallel$ to x-axis.

For Vertical Tangent-

$$\frac{dy}{dx} = \infty$$

$$\Rightarrow \frac{dx}{dt} = 0$$

$$2t = 0$$

$$t = 0$$

Put in (1) and (2)

$$\Rightarrow x = 0 + 4 = 4$$

$$y = 0 - 0 + 2 = 2$$

$\Rightarrow$ at $(4, 2)$ the tangent is vertical.

Find equations of the tangent and Normal to each of the given curves at indicated point. (9-11)

Q#9

Soln.

$$x = 2a \cos \theta - a \cos 2\theta, \quad y = 2a \sin \theta - a \sin 2\theta \text{ at } \theta = \frac{\pi}{2}$$

$$\text{Here } x = 2a \cos \theta - a \cos 2\theta \quad \text{--- (i)}$$

$$y = 2a \sin \theta - a \sin 2\theta \quad \text{--- (ii)}$$

$$\text{I) } \Rightarrow \frac{dx}{d\theta} = -2a \sin \theta + 2a \sin 2\theta$$

$$= 2a (\sin 2\theta - \sin \theta)$$

$$= 2a \cdot 2 \cos \frac{2\theta + \theta}{2} \cdot \sin \frac{2\theta - \theta}{2}$$

$$= 4a \cos \frac{3\theta}{2} \sin \frac{\theta}{2} \quad \text{--- (iii)}$$

$$\text{II) } \Rightarrow \frac{dy}{d\theta} = 2a \cos \theta - 2a \cos 2\theta$$

$$= 2a (\cos \theta - \cos 2\theta)$$

$$= 2a \cdot 2 \sin \frac{\theta + 2\theta}{2} \sin \frac{\theta - 2\theta}{2}$$

$$\begin{aligned}\frac{dy}{d\theta} &= -4a \sin \frac{3\theta}{2} \sin(-\frac{\theta}{2}) \\ &= 4a \sin \frac{3\theta}{2} \sin \frac{\theta}{2}\end{aligned}$$

Now

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{4a \sin \frac{3\theta}{2} \sin \frac{\theta}{2}}{4a \cos \frac{3\theta}{2} \sin \frac{\theta}{2}}$$

$$\frac{dy}{dx} = \frac{\sin \frac{3\theta}{2}}{\cos \frac{3\theta}{2}}$$

$$\frac{dy}{dx} = \tan \frac{3\theta}{2}$$

$$\left. \frac{dy}{dx} \right|_{\pi/2} = \tan \frac{3}{2} \cdot \frac{\pi}{2}$$

$$\left. \frac{dy}{dx} \right|_{\pi/2} = \tan \frac{3\pi}{4} = -1$$

Also $x_1 = 2a \cos \frac{\pi}{2} - a \cos 2 \cdot \frac{\pi}{2} = 0 - a(-1) = a$

$$y_1 = 2a \sin \frac{\pi}{2} - a \sin 2 \cdot \frac{\pi}{2} = 2a - 0 = 2a$$

Thus the eq. of the tangent is

$$y - y_1 = \left. \frac{dy}{dx} \right|_p (x - x_1)$$

$$y - 2a = -1(x - a)$$

$$y - 2a = -x + a$$

$$x + y - 2a - a = 0$$

$$x + y - 3a = 0$$

IN the eq. of the tangent at $\pi/2$

Now eq. of the normal

$$y - y_1 = -\frac{1}{dy/dx} (x - x_1)$$

$$y - 2a = -\frac{1}{-1} (x - a)$$

$$y - 2a = x - a$$

$$x - y - a + 2a = 0$$

$$x - y + a = 0$$

Is the eq. of the tangent at $\pi/2$

③ #10.
soln:

$$x = \frac{2at^2}{1+t^2}, \quad y = \frac{2at^3}{1+t^2} \quad \text{at } t = \frac{1}{2}$$

$$x = \frac{2at^2}{1+t^2} \quad \text{--- (1)}$$

$$y = \frac{2at^3}{1+t^2} \quad \text{--- (2)}$$

$$1) \Rightarrow \frac{dx}{dt} = \frac{(1+t^2) \cdot 4at - 2at^2 \cdot 2t}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{4at - 4at^3 - 4at^3}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{4a(1-t^4)}{(1+t^2)^2} \quad \text{--- (3)}$$

$$2) \Rightarrow \frac{dy}{dt} = \frac{(1+t^2) \cdot 6at^2 - 2at^3 \cdot 2t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{6at^2 + 6at^4 - 4at^4}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{6at^2 + 2at^4}{(1+t^2)^2}$$

$$= \frac{2a(3t^2 + t^4)}{(1+t^2)^2}$$

Now

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2a(3t^2 + t^4)}{(1+t^2)^2} \cdot \frac{(1+t^2)^2}{4a(1-t^4)}$$

$$\frac{dy}{dx} = \frac{2a(3t + t^3)}{4a(1-t^4)}$$

$$\frac{dy}{dx} = \frac{3t + t^3}{2}$$

$$\left. \frac{dy}{dx} \right|_{t=\frac{1}{2}} = \frac{3(\frac{1}{2}) + (\frac{1}{2})^3}{2} = \frac{\frac{3}{2} + \frac{1}{8}}{2} = \frac{\frac{12+1}{8}}{2}$$

$$= \frac{13}{16}$$

Now

$$\begin{aligned}
 x_1 &= \frac{2a(1/2)^2}{1+(1/2)^2}, & y_1 &= \frac{2a(1/2)^3}{1+(1/2)^2} \\
 x_1 &= \frac{2a \cdot 1/4}{1+1/4}, & y_1 &= \frac{2a(1/8)}{1+1/4} \\
 x_1 &= \frac{a/2}{5/4}, & y_1 &= \frac{a/4}{5/4} \\
 x_1 &= \frac{2a}{5}, & y_1 &= \frac{a}{5} \\
 \Rightarrow (x_1, y_1) &= \left(\frac{2a}{5}, \frac{a}{5}\right)
 \end{aligned}$$

So Eq. of the tangent at $t = \frac{1}{2}$ is

$$\begin{aligned}
 y - \frac{a}{5} &= \frac{13}{16} \left(x - \frac{2a}{5}\right) \\
 16y - \frac{16a}{5} &= 13 \left(x - \frac{2a}{5}\right)
 \end{aligned}$$

$$80y - 16a = 65x - 26a$$

$$65x - 80y - 10a = 0$$

$$13x - 16y - 2a = 0$$

which is the eq. of tangent.

Now Eq. of the Normal is

$$y - \frac{a}{5} = -\frac{16}{13} \left(x - \frac{2a}{5}\right)$$

$$65y - 13a = -80x + 32a$$

$$80x + 65y - 45a = 0$$

$$16x + 13y - 9a = 0$$

which is the eq. of the Normal at $t = \frac{1}{2}$

Q# 21.

Here $x = (t-1)^{3/2}$, $y = 3t$ at $t=5$

$$x = (t-1)^{3/2} \quad \text{--- (I)}$$

$$y = 3t \quad \text{--- (II)}$$

Now 1) $\Rightarrow$

$$\frac{dx}{dt} = \frac{3}{2} (t-1)^{1/2}$$

2) $\Rightarrow$

$$\frac{dy}{dt} = 3$$

Now $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{3}{\frac{3}{2}\sqrt{t-1}} = \frac{2}{\sqrt{t-1}}$

$$\left. \frac{dy}{dx} \right|_{t=5} = \frac{2}{\sqrt{5-1}} = \frac{2}{\sqrt{4}} = \frac{2}{2} = 1$$

The point $P(x_1, y_1)$ at $t = 5$

$$x_1 = (5-1)^{3/2} \\ = (4)^{3/2} = 8$$

and $y_1 = 3(5)$

$$y_1 = 15$$

Hence the point $P(x_1, y_1)$ at $t = 5$ is $(8, 15)$

So the Eq. of the tangent is given by

$$y - 15 = 1(x - 8)$$

$$x - y + 7 = 0$$

Eq. of the Normal is

$$y - 15 = -1(x - 8)$$

$$x + y - 23 = 0$$

Q.12 Show that the Normal at any point of the curve $x = a \cos \theta + a \theta \sin \theta$

$$y = a \sin \theta - a \theta \cos \theta$$

is at a constant distance from the origin.

Soln: Here $x = a \cos \theta + a \theta \sin \theta$ _____ (i)

$$y = a \sin \theta - a \theta \cos \theta$$
 _____ (ii)

$$1) \Rightarrow \frac{dx}{d\theta} = -a \sin \theta + a \theta \cos \theta + a \sin \theta \\ = a \theta \cos \theta$$

$$2) \Rightarrow \frac{dy}{d\theta} = a \cos \theta + a \theta \sin \theta - a \cos \theta \\ = a \theta \sin \theta$$

$$\text{But } \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a \theta \sin \theta}{a \theta \cos \theta} = \tan \theta$$

Now the Eq. of the Normal at $(a \cos \theta + a \theta \sin \theta, a \sin \theta - a \theta \cos \theta)$

$$y - y_1 = -\frac{1}{dy/dx} (x - x_1)$$

$$y - (a \sin \theta - a \theta \cos \theta) = -\frac{1}{\tan \theta} (x - (a \cos \theta + a \theta \sin \theta))$$

$$y - a \sin \theta + a \theta \cos \theta = - \frac{\cos \theta}{\sin \theta} [x - a \cos \theta - a \theta \sin \theta]$$

x by $\sin \theta$

$$y \sin \theta - a \sin^2 \theta + a \theta \sin \theta \cos \theta = -x \cos \theta + a \cos^2 \theta + a \theta \sin \theta \cos \theta$$

$$\Rightarrow x \cos \theta + y \sin \theta - a \sin^2 \theta - a \cos^2 \theta = 0$$

$$x \cos \theta + y \sin \theta - a (\sin^2 \theta + \cos^2 \theta) = 0$$

$$x \cos \theta + y \sin \theta - a = 0 \quad \text{--- iii}$$

which is the eq. of Normal

Now If d is the distance of this normal from $O(0,0)$ then

$$d = \frac{|0 + 0 - a|}{\sqrt{\cos^2 \theta + \sin^2 \theta}}$$

$$d = \frac{a}{1} = a$$

= Constant.

Q #13. Prove that an equation of the Normal to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ can be written in the form

$$x \sin t - y \cos t + a \cos 2t = 0, \quad t \text{ being parameter.}$$

Soln: The parametric eq. of the astroid is

$$x = a \cos^3 t \quad (1) \quad \text{and} \quad y = a \sin^3 t \quad (2)$$

$$1) \Rightarrow \frac{dx}{dt} = -3a \sin^2 t \cos t$$

$$2) \Rightarrow \frac{dy}{dt} = 3a \cos^2 t \sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = - \frac{3a \cos^2 t \sin t}{3a \sin^2 t \cos t}$$

$$= - \frac{\cos t}{\sin t}$$

So eq. of the Normal at $(a \cos^3 t, a \sin^3 t)$

$$\text{is} \quad y - a \sin^3 t = \frac{\sin t}{\cos t} (x - a \cos^3 t)$$

$$\Rightarrow y \cos t - a \cos^4 t = x \sin t - a \sin^4 t$$

$$\Rightarrow x \sin t - y \cos t + a \cos^4 t - a \sin^4 t = 0$$

$$x \sin t - y \cos t + a (\cos^4 t - \sin^4 t) = 0$$

$$x \sin t - y \cos t + a [(\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t)] = 0$$

$$x \sin t - y \cos t + a (\cos 2t) = 0$$

$$x \sin t - y \cos t + a \cos 2t = 0$$

which is the required eq. of the Normal.

Q#14. Show that the pedal eq. of the astroid

$$x = a \cos^3 \theta, \quad y = a \sin^3 \theta$$

is $r^2 = a^2 - 3p^2$

Soln.

Here $x = a \cos^3 \theta$ _____ (1)

$y = a \sin^3 \theta$ _____ (2)

1) $\Rightarrow \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$

2) $\Rightarrow \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$

But $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = - \frac{3a \sin^2 \theta \cos \theta}{3a \cos^2 \theta \sin \theta}$
 $= - \frac{\sin \theta}{\cos \theta}$

Now Eq. of the tangent at $(a \cos^3 \theta, a \sin^3 \theta)$ is given by

$$y - a \sin^3 \theta = - \frac{\sin \theta}{\cos \theta} (x - a \cos^3 \theta)$$

$$y \cos \theta - a \sin^3 \theta \cos \theta = -x \sin \theta + a \cos^3 \theta \sin \theta$$

$$\sin \theta x + \cos \theta y - a [\sin^3 \theta \cos \theta + \cos^3 \theta \sin \theta] = 0$$

$$\sin \theta x + \cos \theta y - a \sin \theta \cos \theta (\sin^2 \theta + \cos^2 \theta) = 0$$

$$\sin \theta x + \cos \theta y - a \sin \theta \cos \theta = 0$$

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{|0 + 0 - a \sin \theta \cos \theta|}{\sqrt{\sin^2 \theta + \cos^2 \theta}}$$

$$p = a \sin \theta \cos \theta$$

$$\frac{p}{a} = \sin \theta \cos \theta \quad \text{--- (3)}$$

$$r^2 = x^2 + y^2$$

$$r^2 = a^2 \cos^2 \theta + a^2 \sin^2 \theta$$

$$r^2 = a^2 (\cos^2 \theta + \sin^2 \theta)$$

$$r^2 = a^2 [(\cos^2 \theta)^3 + (\sin^2 \theta)^3]$$

$$r^2 = a^2 [(\cos^2 \theta + \sin^2 \theta)^3 - 3 (\cos^2 \theta \sin^2 \theta) (\cos^2 \theta + \sin^2 \theta)]$$

$$r^2 = a^2 [1 - 3 \cos^2 \theta \sin^2 \theta] \quad \text{--- (4)}$$

Put (3) in (4)

$$r^2 = a^2 \left[1 - 3 \frac{p^2}{a^2} \right]$$

$$\Rightarrow r^2 = a^2 - 3p^2$$

Is the required pedal Equation.

Q#15

Prove that the pedal Eq. of the curve

$$x = 2a \cos \theta - a \cos 2\theta, \quad y = 2a \sin \theta - a \sin 2\theta$$

$$\text{is } 9(r^2 - a^2) = 8p^2$$

Soln:

Here

$$x = 2a \cos \theta - a \cos 2\theta \quad \text{--- I}$$

$$y = 2a \sin \theta - a \sin 2\theta \quad \text{--- II}$$

$$\text{I) } \Rightarrow \frac{dx}{d\theta} = -2a \sin \theta + 2a \sin 2\theta$$

$$= 2a [\sin 2\theta - \sin \theta]$$

$$\text{II) } \Rightarrow \frac{dy}{d\theta} = 2a \cos \theta - 2a \cos 2\theta$$

$$= 2a [\cos \theta - \cos 2\theta]$$

$$\text{But } \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{2a [\cos \theta - \cos 2\theta]}{2a [\sin 2\theta - \sin \theta]}$$

$$= \frac{-2 \sin \frac{\theta+2\theta}{2} \sin \frac{\theta-2\theta}{2}}{2 \cos \frac{2\theta+\theta}{2} \sin \frac{2\theta-\theta}{2}}$$

$$= \frac{\sin 3\theta/2 \cdot \sin (-\theta/2)}{\cos 3\theta/2 \cdot \sin \theta/2}$$

$$= - \frac{\sin 3\theta/2 \cdot \sin (-\theta/2)}{\cos 3\theta/2 \cdot \sin \theta/2}$$

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$

$$\frac{dy}{dx} = \frac{\sin 3\theta/2}{\cos 3\theta/2}$$

Eq. of the tangent at $(2a \cos \theta - a \cos 2\theta, 2a \sin \theta - a \sin 2\theta)$ is
 $y - 2a \sin \theta + a \sin 2\theta = \frac{\sin 3\theta/2}{\cos 3\theta/2} (x - 2a \cos \theta + a \cos 2\theta)$

$$\Rightarrow \cos \frac{3\theta}{2} y - 2a \sin \theta \cos \frac{3\theta}{2} + a \sin 2\theta \cos \frac{3\theta}{2} = x \sin \frac{3\theta}{2} - 2a \cos \theta \sin \frac{3\theta}{2} + a \cos 2\theta \sin \frac{3\theta}{2}$$

$$\Rightarrow x \sin \frac{3\theta}{2} - y \cos \frac{3\theta}{2} - 2a \left[\sin \frac{3\theta}{2} \cos \theta - \sin \theta \cos \frac{3\theta}{2} \right] + a \left[\sin \frac{3\theta}{2} \cos 2\theta - \sin 2\theta \cos \frac{3\theta}{2} \right] = 0$$

$$\Rightarrow x \sin \frac{3\theta}{2} - y \cos \frac{3\theta}{2} - 2a \left[\sin \left(\frac{3\theta}{2} - \theta \right) \right] + a \left[\sin \left(\frac{3\theta}{2} - 2\theta \right) \right] = 0$$

$$\Rightarrow x \sin \frac{3\theta}{2} - y \cos \frac{3\theta}{2} - 2a \sin \frac{\theta}{2} - a \sin \frac{\theta}{2} = 0$$

$$\Rightarrow x \sin \frac{3\theta}{2} - y \cos \frac{3\theta}{2} - 3a \sin \frac{\theta}{2} = 0$$

which is the equation of tangent.

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{|0 - 0 - 3a \sin \frac{\theta}{2}|}{\sqrt{\sin^2 \frac{3\theta}{2} + \cos^2 \frac{3\theta}{2}}}$$

$$p = 3a \sin \frac{\theta}{2} \quad \text{--- III}$$

Now

$$r^2 = x^2 + y^2$$

$$r^2 = (2a \cos \theta - a \cos 2\theta)^2 + (2a \sin \theta - a \sin 2\theta)^2$$

$$r^2 = 4a^2 \cos^2 \theta + a^2 \cos^2 2\theta - 4a^2 \cos \theta \cos 2\theta + 4a^2 \sin^2 \theta + a^2 \sin^2 2\theta - 4a^2 \sin \theta \sin 2\theta$$

$$r^2 = 4a^2 (\cos^2 \theta + \sin^2 \theta) + a^2 (\cos^2 2\theta + \sin^2 2\theta)$$

$$- 4a^2 [\cos 2\theta \cos \theta + \sin 2\theta \sin \theta]$$

$$r^2 = 4a^2 + a^2 - 4a^2 \cos(2\theta - \theta)$$

$$r^2 = 5a^2 - 4a^2 \cos \theta$$

$$r^2 = 5a^2 - 4a^2 \left[1 - 2 \sin^2 \frac{\theta}{2} \right]$$

$$r^2 = 5a^2 - 4a^2 + 8a^2 \sin^2 \frac{\theta}{2}$$

$$r^2 = a^2 + 8a^2 \sin^2 \frac{\theta}{2}$$

$$r^2 - a^2 = 8a^2 \sin^2 \frac{\theta}{2} \quad \text{--- IV}$$

Put (3) in (4)

$$\Rightarrow r^2 - a^2 = \frac{8c^2 b^2}{9a^2}$$

$$9(r^2 - a^2) = 8b^2$$

which is the required pedal Equation.

Q#16. Show that the pedal Equation of the curve
 $x = ae^{\theta}(\sin \theta - \cos \theta)$, $y = ae^{\theta}(\sin \theta + \cos \theta)$
 is $r = \sqrt{2} p$

Soln.

$$\text{Here } x = ae^{\theta}(\sin \theta - \cos \theta) \quad \text{--- (1)}$$

$$y = ae^{\theta}(\sin \theta + \cos \theta) \quad \text{--- (2)}$$

$$\begin{aligned} 1) \Rightarrow \frac{dx}{d\theta} &= ae^{\theta}(\sin \theta - \cos \theta) + ae^{\theta}(\cos \theta + \sin \theta) \\ &= ae^{\theta} \sin \theta - ae^{\theta} \cos \theta + ae^{\theta} \cos \theta + ae^{\theta} \sin \theta \\ &= 2ae^{\theta} \sin \theta \end{aligned}$$

$$\begin{aligned} 2) \Rightarrow \frac{dy}{d\theta} &= ae^{\theta}(\sin \theta + \cos \theta) + ae^{\theta}(\cos \theta - \sin \theta) \\ &= ae^{\theta} \sin \theta + ae^{\theta} \cos \theta + ae^{\theta} \cos \theta - ae^{\theta} \sin \theta \\ &= 2ae^{\theta} \cos \theta \end{aligned}$$

$$\begin{aligned} \text{But } \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2ae^{\theta} \cos \theta}{2ae^{\theta} \sin \theta} \\ &= \frac{\cos \theta}{\sin \theta} \end{aligned}$$

Eq. of the tangent at $(ae^{\theta}(\sin \theta - \cos \theta), ae^{\theta}(\sin \theta + \cos \theta))$ is

$$y - ae^{\theta} \sin \theta - ae^{\theta} \cos \theta = \frac{\cos \theta}{\sin \theta} (x - ae^{\theta} \sin \theta + ae^{\theta} \cos \theta)$$

$$\begin{aligned} y \sin \theta - ae^{\theta} \sin^2 \theta - ae^{\theta} \sin \theta \cos \theta &= x \cos \theta - ae^{\theta} \sin \theta \cos \theta + ae^{\theta} \cos^2 \theta \\ x \cos \theta - y \sin \theta + ae^{\theta} (\cos^2 \theta + \sin^2 \theta) &= 0 \end{aligned}$$

$$\begin{aligned} 1 - \cos \theta &= 2 \sin^2 \frac{\theta}{2} \\ \Rightarrow 1 - 2 \sin^2 \frac{\theta}{2} &= \cos \theta \end{aligned}$$

$$x \cos \theta - y \sin \theta - a e^{\theta} = 0$$

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{|0 - 0 - a e^{\theta}|}{\sqrt{\cos^2 \theta + \sin^2 \theta}}$$

$$p = a e^{\theta} \quad \text{--- III}$$

$$\therefore r^2 = x^2 + y^2$$

$$r^2 = a^2 e^{2\theta} \{ (\sin \theta - \cos \theta)^2 + (\sin \theta + \cos \theta)^2 \}$$

$$r^2 = a^2 e^{2\theta} [\sin^2 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta + \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta]$$

$$r^2 = 2 a^2 e^{2\theta} \quad \text{--- IV}$$

Put III in IV

$$r^2 = 2 p^2$$

$$r = \sqrt{2} p$$

which is the required pedal Equation.

Q#17. Prove that the pedal Equation of the Curve

$$x = a(3 \cos \theta - \cos^3 \theta), \quad y = a(3 \sin \theta - \sin^3 \theta)$$

$$\text{is } 3p^2(7a^2 - r^2) = (10a^2 - r^2)^2.$$

Soln.

Here

$$x = a(3 \cos \theta - \cos^3 \theta) \quad \text{--- (1)}$$

$$y = a(3 \sin \theta - \sin^3 \theta) \quad \text{--- (2)}$$

$$\begin{aligned} 1) \Rightarrow \frac{dx}{d\theta} &= -3a \sin \theta + 3a \cos^2 \theta \sin \theta \\ &= 3a \sin \theta [\cos^2 \theta - 1] \\ &= 3a \sin \theta [-\sin^2 \theta] \\ &= -3a \sin^3 \theta \end{aligned}$$

$$\begin{aligned} 2) \Rightarrow \frac{dy}{d\theta} &= 3a \cos \theta - 3a \sin^2 \theta \cos \theta \\ &= 3a \cos \theta [1 - \sin^2 \theta] \\ &= 3a \cos \theta [\cos^2 \theta] \\ &= 3a \cos^3 \theta \end{aligned}$$

$$\text{But } \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \frac{3a \cos^3 \theta}{-3a \sin^3 \theta}$$

$$\frac{dy}{dx} = - \frac{\cos^3 \theta}{\sin^3 \theta}$$

Eq. of the tangent at $(a(3\cos\theta - \cos^3\theta), a(3\sin\theta - \sin^3\theta))$

$$y - 3a\sin\theta + a\sin^3\theta = - \frac{\cos^3\theta}{\sin^3\theta} (x - 3a\cos\theta + a\cos^3\theta)$$

$$\begin{aligned} y\sin^3\theta - 3a\sin^4\theta + a\sin^6\theta &= -x\cos^3\theta + 3a\cos^4\theta - a\cos^6\theta \\ x\cos^3\theta + y\sin^3\theta + a[\sin^6\theta + \cos^6\theta] &= 3a[\sin^4\theta + \cos^4\theta] \\ x\cos^3\theta + y\sin^3\theta + a[(\sin^2\theta)^3 + (\cos^2\theta)^3] &= 3a[(\sin^2\theta)^2 + (\cos^2\theta)^2] \\ x\cos^3\theta + y\sin^3\theta + a[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] &= 3a[(\sin^2\theta + \cos^2\theta)^2 - 2\cos^2\theta\sin^2\theta] \\ x\cos^3\theta + y\sin^3\theta + a[1 - 3\sin^2\theta\cos^2\theta] &= 3a[1 - 2\cos^2\theta\sin^2\theta] \\ x\cos^3\theta + y\sin^3\theta + a - 3a\sin^2\theta\cos^2\theta &= 3a - 6a\sin^2\theta\cos^2\theta \\ x\cos^3\theta + y\sin^3\theta + a - 3a - 3a\sin^2\theta\cos^2\theta + 6a\sin^2\theta\cos^2\theta &= 0 \\ x\cos^3\theta + y\sin^3\theta - 2a + 3a\sin^2\theta\cos^2\theta &= 0 \\ x\cos^3\theta + y\sin^3\theta + 3a\sin^2\theta\cos^2\theta - 2a &= 0 \end{aligned}$$

$$\begin{aligned} a^2 + b^2 &= (a+b)^2 - 2ab \\ a^3 + b^3 &= (a+b)^3 - 3ab(a+b) \end{aligned}$$

which is the Eq. of the tangent.

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{|0 + 0 + 3a\sin^2\theta\cos^2\theta - 2a|}{\sqrt{\cos^6\theta + \sin^6\theta}}$$

$$p = \frac{|3a\sin^2\theta\cos^2\theta - 2a|}{\sqrt{1 - 3\sin^2\theta\cos^2\theta}} \quad \text{--- (3)}$$

Now $r^2 = x^2 + y^2$

$$r^2 = a^2(3\cos\theta - \cos^3\theta)^2 + a^2(3\sin\theta - \sin^3\theta)^2$$

$$r^2 = a^2(9\cos^2\theta + \cos^6\theta - 6\cos^4\theta) + a^2(9\sin^2\theta + \sin^6\theta - 6\sin^4\theta)$$

$$r^2 = 9a^2(\cos^2\theta + \sin^2\theta) + a^2 6(-\cos^4\theta - \sin^4\theta) + a^2(\sin^6\theta + \cos^6\theta)$$

$$r^2 = a^2[9 - 6(\cos^4\theta + \sin^4\theta) + (\sin^6\theta + \cos^6\theta)]$$

$$r^2 = a^2[9 - 6\{(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta\} + (\sin^2\theta + \cos^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\sin^2\theta + \cos^2\theta)]$$

$$r^2 = a^2 [9 - 6 \{1 - 2 \sin^2 \theta \cos^2 \theta\} + 1 - 3 \sin^2 \theta \cos^2 \theta]$$

$$r^2 = 9a^2 - 6a^2 + 12a^2 \sin^2 \theta \cos^2 \theta + a^2 - 3a^2 \sin^2 \theta \cos^2 \theta$$

$$r^2 = 4a^2 + 9a^2 \cos^2 \theta \sin^2 \theta$$

$$r^2 = a^2 [4 + 9 \cos^2 \theta \sin^2 \theta]$$

$$r^2 - 4a^2 = 9a^2 \cos^2 \theta \sin^2 \theta$$

$$\Rightarrow \cos^2 \theta \sin^2 \theta = \frac{r^2 - 4a^2}{9a^2} \quad \text{--- (4)}$$

Put (4) in (3)

$$p = \frac{|3a \cdot \frac{r^2 - 4a^2}{9a^2} - 2a|}{\sqrt{1 - 3 \left[\frac{r^2 - 4a^2}{9a^2} \right]}}$$

$$p = \frac{\left| \frac{r^2 - 4a^2}{3a} - 2a \right|}{\sqrt{1 - \left(\frac{r^2 - 4a^2}{3a^2} \right)}}$$

$$p = \frac{r^2 - 4a^2 - 6a^2}{3a} \cdot \frac{\sqrt{3} a}{\sqrt{3a^2 - r^2 + 4a^2}}$$

$$p = \frac{r^2 - 10a^2}{\sqrt{3}} \times \frac{1}{\sqrt{7a^2 - r^2}}$$

$$p = \frac{r^2 - 10a^2}{\sqrt{3} \sqrt{7a^2 - r^2}}$$

$$\Rightarrow p^2 = \frac{(r^2 - 10a^2)^2}{3(7a^2 - r^2)}$$

$$3p^2(7a^2 - r^2) = (r^2 - 10a^2)^2$$

is the required pedal equation.

Q #18. If $x = a \cos g(t)$, $y = b \sin g(t)$, prove that

$$xy^2 \frac{d^2 y}{dx^2} = b^2 \frac{dy}{dx}$$

Soln:

$$\text{Here } x = a \cos g(t) \quad \text{--- I}$$

$$y = b \sin g(t) \quad \text{--- II}$$

$$\Rightarrow \frac{dx}{dt} = -a \sin g(t) \cdot g'(t)$$

$$\Rightarrow \frac{dy}{dt} = b \cos g(t) \cdot g'(t)$$

$$\frac{dy}{dx} = \frac{b \cos g(t) \cdot g'(t)}{-a \sin g(t) \cdot g'(t)}$$

$$= -\frac{b}{a} \cot g(t) \quad \text{--- (iii)}$$

Again) Diff. w.r.t. 'x'

$$\frac{d^2y}{dx^2} = \frac{b}{a} \operatorname{cosec}^2 g(t) \cdot g'(t) \cdot \frac{dt}{dx}$$

$$= \frac{b}{a} \operatorname{cosec}^2 g(t) \cdot g'(t) \cdot \frac{1}{-a \sin g(t) \cdot g'(t)}$$

$$= -\frac{b}{a^2 \sin^3 g(t)}$$

Consider

$$xy^2 \frac{d^2y}{dx^2} = a \cos g(t) \cdot b^2 \sin^4 g(t) \cdot -\frac{b}{a^2 \sin^3 g(t)}$$

$$= -\frac{b^3 \cos g(t)}{a \sin g(t)}$$

$$= b^2 \left(-\frac{b \cos g(t)}{a \sin g(t)} \right)$$

$$= b^2 \frac{dy}{dx}$$

from (iii)

$$\text{Hence } xy^2 \frac{d^2y}{dx^2} = b^2 \frac{dy}{dx}$$

is as required.