

Exercise # 1.3

Discuss the continuity of the following functions at the indicated points/set.

1. $f(x) = |x-3|$ at $x=3$

$$\begin{aligned}\lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} |x-3| \\ &= \lim_{\substack{x=3-h \\ h \rightarrow 0}} |3-h-3| = \lim_{h \rightarrow 0} |-h| = \lim_{h \rightarrow 0} h\end{aligned}$$

$$\lim_{h \rightarrow 0} f(x) = 0$$

$$\begin{aligned}\lim_{x \rightarrow 3^+} f(x) &= \lim_{x \rightarrow 3^+} |x-3| = \lim_{\substack{x=3+h \\ h \rightarrow 0}} |3+h-3| \\ &= \lim_{h \rightarrow 0} |h| = \lim_{h \rightarrow 0} h = 0\end{aligned}$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3} f(x)$$

$$f(3) = |3-3| = 0$$

$$\lim_{x \rightarrow 3} f(x) = f(3)$$

hence function is continuous at $x=3$.

$$2. \quad f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 0 & \text{if } x=3 \end{cases}$$

Sol:

$$f(3) = 0$$

$$\begin{aligned}\lim_{x \rightarrow 3} f(x) &= \lim_{x \rightarrow 3} \frac{x^2-9}{x-3} = \lim_{x \rightarrow 3} \frac{(x+3)(x-3)}{(x-3)} \\ &= \lim_{x \rightarrow 3} (x+3) = 3+3\end{aligned}$$

$$\lim_{x \rightarrow 3} f(x) = 6$$

$$\text{So } \lim_{x \rightarrow 3} f(x) \neq f(3)$$

function is discontinuous.

3. $f(x) = \begin{cases} x-4 & \text{if } -1 \leq x \leq 2 \\ x^2-6 & \text{if } 2 < x < 5 \end{cases}$ at $x=2$

Sol.

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x-4) = 2-4$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2-6) = 4-6$$

$$\lim_{x \rightarrow 2^-} f(x) = -2$$

$$\lim_{x \rightarrow 2^+} f(x) = -2$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2} f(x) = -2$$

$$f(2) = x-4 = 2-4 = -2$$

$$\lim_{x \rightarrow 2} f(x) = f(2)$$

Function is continuous.

4. $f(x) = \begin{cases} \frac{x^3-27}{x^2-9} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ at $x=3$

Sol.

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{x^3-27}{x^2-9}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)}$$

$$= \lim_{x \rightarrow 3} \frac{(x^2+3x+9)}{x+3}$$

$$= \frac{9+9+9}{3+3}$$

$$= \frac{27}{6}$$

$$\lim_{x \rightarrow 3} f(x) = \frac{9}{2}$$

$$f(3) = 6$$

So here $\lim_{x \rightarrow 3} f(x) \neq f(3)$

function is discontinuous.

5. $f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ at $x=0$.

Sol.

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \sin \frac{1}{x} = \sin(-\infty) = \text{any value b/w } [-1, 1]$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sin \left(\frac{1}{x} \right) = \sin(\infty) = \text{any value b/w } [-1, 1]$$

Limit does not exist.

Function is discontinuous.

6. $f(x) = \sin x \quad \forall x \in \mathbb{R}$.

Let θ be an arbitrary real no.

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \sin x = \sin 0.$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sin x = \sin 0$$

$$f(0) = \sin 0$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(x) = f(0)$$

$\Rightarrow f(x)$ is continuous at $x=0$

$\therefore 0 \in \mathbb{R} \Rightarrow f(x)$ is continuous $\forall \mathbb{R}$.

$$7. \quad f(x) = \begin{cases} \frac{x^2 - a}{a} & \text{if } 0 < x < a \\ 0 & \text{if } x = a \\ a - \frac{a^2}{x} & \text{if } x > a. \end{cases} \quad \text{at } x=a$$

Sol:

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} \left(\frac{x^2 - a}{a} \right)$$

$$= \frac{a^2 - a}{a}$$

$$= a - a$$

$$\lim_{x \rightarrow a^-} f(x) = 0$$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} \left(a - \frac{a^2}{x} \right)$$

$$= a - \frac{a^2}{a}$$

$$= a - a$$

$$\lim_{x \rightarrow a^+} f(x) = 0$$

$$\Rightarrow \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a} f(x) = 0$$

$$f(a) = 0$$

$$\lim_{x \rightarrow a} f(x) = f(a) = 0$$

function is continuous at $x=a$

8. Determine all the points of continuity of the function $f(x) = x - [x] \quad \forall x \in \mathbb{R}$.

Sol. $[...]$ being Bracket function.

$$\text{when } 0 \leq x < 1 \quad [x] = 0$$

$$\text{when } 1 \leq x < 2 \quad [x] = 1$$

$$\text{when } 2 \leq x < 3 \quad [x] = 2$$

$\vdots$

$$\text{when } -1 < x < 0 \quad [x] = -1$$

$$\text{when } -2 < x < -1 \quad [x] = -2$$

$$\text{when } -3 < x < -2 \quad [x] = -3$$

$\vdots$

$$\Rightarrow f(x) = x \quad 0 < x < 1$$

$$f(x) = x - 1 \quad 1 < x < 2$$

$$f(x) = x - 2 \quad 2 < x < 3$$

$\vdots$

$$f(x) = x + 1 \quad -1 < x < 0$$

$$f(x) = x + 2 \quad -2 < x < -1$$

$$f(x) = x + 3 \quad -3 < x < -2$$

At $x=1$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x) = 1$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x-1) = 0$$

Therefore, Limit does not exist. Function is not continuous at $x=1$.

At $x=0$,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x+1) = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x) = 0$$

Limit does not exist, function is not continuous at $x=0$

Similarly, it is continuous for integral values of x , both positive and negative, but it is continuous at every other real value of x .

9. Discuss continuity at $x=1$, $f(x) = x - |x|$.

Sol.

$$f(x) = x - |x|$$

$$\lim_{x \rightarrow 1^-} x - |x| = 1 - 1 = 0$$

$$\lim_{x \rightarrow 1^+} x - |x| = 1 - 1 = 0$$

$$f(1) = 1 - 1 = 0$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = f(1)$$

$\Rightarrow$ function is continuous at $x=1$.

10. Show that function $f: \mathbb{R} \rightarrow \mathbb{R}$, defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational.} \\ 1-x & \text{if } x \text{ is rational.} \end{cases} \text{ is cont. at } x = \frac{1}{2}.$$

Sol. $f\left(\frac{1}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

$$\lim_{x \rightarrow 1/2} f(x) = \lim_{x \rightarrow 1/2} (1-x) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow 1/2} f(x) = f(1/2)$$

function is continuous.

11. Show that function $f: [0, 1] \rightarrow \mathbb{R}$, defined by $f(x) = \frac{1}{x}$ is continuous on $[0, 1]$. Is $f(x)$ is bounded on this interval? Explain.

Sol.

$f(x)$ is defined for all real values of x such that $0 < x \leq 1$ and its limit exists at each such x and equals to its value there, so it is

continuous on $[0, 1]$.

When $x = 1$, $f(x) = 1$ which is its lower bound. So it is bounded below. $f(x)$ increases indefinitely as x becomes small. Thus $f(x)$ is not bounded above. Hence $f(x)$ is not bounded on $[0, 1]$.

12. Let $f(x) = \begin{cases} \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, Is f continuous at $x = 0$.

Sol. $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \cos(\frac{1}{x}) = \text{any value} \in [-1, 1]$

Thus Limit does not exist. Similarly,

$\lim_{x \rightarrow 0^+} f(x)$ does not exist.

Hence $\lim_{x \rightarrow 0} f(x)$ does not exist.

So given function is not continuous at $x = 0$.

13. Let $f(x) = \begin{cases} (x-a) \sin(\frac{1}{x-a}) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$

Discuss continuity of 'f' at $x = a$.

Sol. $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} (x-a) \sin(\frac{1}{x-a})$
 $= \lim_{x \rightarrow a} (x-a) \lim_{x \rightarrow a} \sin(\frac{1}{x-a})$
 $= (a-a) \cdot \sin(\frac{1}{a-a})$
 $= 0 \cdot \sin(\frac{1}{0})$
 $= 0 \cdot (\text{any value from } [-1, 1])$

$\lim_{x \rightarrow a} f(x) = 0$

$f(a) = 0$

By defined function

$\lim_{x \rightarrow a} f(x) = f(a) = 0$

So given f is continuous at $x = a$.

14. Let $f(x) = \begin{cases} x \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Show f is continuous at $x = 0$

Sol. $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x \cos\left(\frac{1}{x}\right)$ $f(0) = 0$

$$= \lim_{x \rightarrow 0} x \cdot \lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$$

$$= 0 \cdot \cos\left(\frac{1}{0}\right)$$

$$= 0 \cdot [\text{any value b/w } [-1, 1]]$$

$$\lim_{x \rightarrow 0} f(x) = 0$$

So $\lim_{x \rightarrow 0} f(x) = f(0)$

So given function is continuous at $x=0$.

15. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ Discuss cont. of f at $x=0$.

Sol. $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$ $f(0) = 0$

$$= \lim_{x \rightarrow 0} x^2 \cdot \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$$

$$= 0 \cdot \sin\left(\frac{1}{0}\right)$$

$$= 0 [\text{any value b/w } [-1, 1]]$$

$$\lim_{x \rightarrow 0} f(x) = 0$$

So $\lim_{x \rightarrow 0} f(x) = f(0) = 0$

So $f(x)$ is continuous at $x=0$

16. Let $f(x) = \begin{cases} x \sin \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ Discuss continuity.

Sol. $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x \sin \frac{|x|}{x} = \lim_{x \rightarrow 0^-} x \cdot \lim_{x \rightarrow 0^-} \frac{\sin |x|}{x}$

$$= 0 \lim_{x \rightarrow 0^-} \sin \frac{-x}{x} = 0 \sin(-1) = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = 0$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x \sin \frac{|x|}{x} = \lim_{x \rightarrow 0^+} x \cdot \lim_{x \rightarrow 0^+} \frac{\sin |x|}{x}$$

$$= 0 \cdot \sin \frac{x}{x} = 0 \sin(1) = 0$$

$$\lim_{x \rightarrow 0^+} f(x) = 0 = \lim_{x \rightarrow 0^-} f(x)$$

$$\lim_{x \rightarrow 0} f(x) = 0$$

So $\lim_{x \rightarrow 0} f(x) = f(0)$

Function is continuous at $x=0$

17. Find c , such that the function $f(x) =$

$$f(x) = \begin{cases} \frac{1-\sqrt{x}}{x-1} & \text{if } 0 \leq x < 1 \\ c & \text{if } x = 1 \end{cases}, \text{ is continuous at } x=1 \forall x \in [0, 1]$$

Sol.

Let $a \in [0, 1]$: arbitrary 'a'

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \frac{1-\sqrt{x}}{x-1} = \frac{1-\sqrt{a}}{a-1} = f(a)$$

Thus f is continuous at a and $\therefore a$ is arbitrary point of $[0, 1]$. Since f is continuous on $[0, 1]$. So f is continuous on $x=1$

$$\begin{aligned} \lim_{x \rightarrow 1} f(x) &= \lim_{x \rightarrow 1} \frac{(1-\sqrt{x})}{x-1} = \lim_{x \rightarrow 1} \frac{-(\sqrt{x}-1)}{(\sqrt{x})^2 - (1)^2} \\ &= \lim_{x \rightarrow 1} \frac{-(\sqrt{x}-1)}{(\sqrt{x}+1)(\sqrt{x}-1)} = \lim_{x \rightarrow 1} \frac{-1}{\sqrt{x}+1} = \frac{-1}{1+1} \end{aligned}$$

$$\lim_{x \rightarrow 1} f(x) = -\frac{1}{2}$$

$$f(1) = c$$

Since function is continuous So

$$\lim_{x \rightarrow 1} f(x) = f(1) \Rightarrow c = -1/2$$

In Problem 18-20. find the points of discontinuity of the given functions.

$$18. f(x) = \begin{cases} x+4 & \text{if } -6 \leq x < -2 \\ x & \text{if } -2 \leq x < 2 \\ x-4 & \text{if } 2 \leq x < 6 \end{cases}$$

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} (x+4) = -2+4$$

$$\lim_{x \rightarrow -2^-} f(x) = 2$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} x = -2$$

$$\lim_{x \rightarrow -2^-} f(x) \neq \lim_{x \rightarrow -2^+} f(x)$$

Limit does not exist.

So function is discontinuous.

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x) = 2$$

$$\begin{aligned} \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (x-4) \\ &= 2-4 \end{aligned}$$

$$\lim_{x \rightarrow 2^+} f(x) = 2$$

$$\text{So } \lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$$

Function is discontinuous.

19. $g(x) = \begin{cases} x^3 & \text{if } x < 1 \\ -4 - x^2 & \text{if } 1 \leq x \leq 10 \\ 6x^2 + 46 & \text{if } x > 10 \end{cases}$

$$\lim_{x \rightarrow 1^-} g(x) = \lim_{x \rightarrow 1^-} x^3 = 1$$

$$\lim_{x \rightarrow 1^+} g(x) = \lim_{x \rightarrow 1^+} (-4 - x^2) = -4 - 1$$

$$\lim_{x \rightarrow 1^+} g(x) = -5$$

$$\lim_{x \rightarrow 1^-} g(x) \neq \lim_{x \rightarrow 1^+} g(x)$$

function is discontinuous at $x=1$

$$\lim_{x \rightarrow 10^-} g(x) = \lim_{x \rightarrow 10^-} (-4 - x^2) = -4 - 100 = -104$$

$$\lim_{x \rightarrow 10^+} g(x) = \lim_{x \rightarrow 10^+} (6x^2 + 46) = 600 + 46 = 646$$

$$\lim_{x \rightarrow 10^-} g(x) \neq \lim_{x \rightarrow 10^+} g(x)$$

function is discontinuous.

20. $f(x) = \begin{cases} x+2 & \text{if } 0 \leq x < 1 \\ x & \text{if } 1 \leq x < 2 \\ x+5 & \text{if } 2 \leq x < 3 \end{cases}$

Sol. $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x+2) = 1+2 = 3$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x) = 1$$

$$\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$$

function is not continuous.

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x) = 2$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x+5) = 2+5 = 7$$

$$\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$$

Function is not continuous.

21. Find constants 'a' and 'b' $\Rightarrow$ function f defined by $f(x) = \begin{cases} x^3 & \text{if } 0 \leq x < -1 \\ ax+b & \text{if } -1 \leq x < 1 \\ x^2+2 & \text{if } x \geq 1 \end{cases}$ is continuous on all x .

Sol. function is continuous on $x=1$

So Limit exists.

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (ax+b) = a+b$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2+2) = 1+2 = 3$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$$

$$a+b = 3 \rightarrow \textcircled{1}$$

function is continuous on $x=-1$

So Limit exists.

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} x^3 = -1$$

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (ax+b) = -a+b$$

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x)$$

$$-a+b = -1 \rightarrow \textcircled{2}$$

adding

$$\begin{array}{r} \textcircled{1} \quad \Delta \quad \textcircled{2} \\ a+b=3 \\ -a+b=-1 \\ \hline 2b=2 \\ \Rightarrow \boxed{b=1} \end{array}$$

Put value of 'b' in $\textcircled{1}$

$$\begin{array}{l} a+b=3 \\ a+1=3 \\ a=3-1 \\ \Rightarrow \boxed{a=2} \end{array}$$

Find the interval on which the given function is continuous. Also find points where it is discontinuous. (Problem 22-26)

22. $f(x) = \frac{x^2-5}{x-1}$

Sol. function $f(x) = \frac{x^2-5}{x-1}$ is not defined at $x=1$.
Thus function is discontinuous at $x=1$.

Numerator is continuous at every point of $\mathbb{R}$ and so is the denominator is $x-1$.

Hence $f(x)$ is continuous at every point of $\mathbb{R} - \{1\}$.

23. $f(x) = \frac{x}{|x|}$

Sol. $f(x)$ is not defined at $x=0$ so $f(x)$ is discontinuous at $x=0$ and continuous at $\mathbb{R} - \{0\}$.

24. $f(x) = \frac{\sin x}{x}$

Sol. $f(x)$ is not defined at $x=0$. Hence it is discontinuous at $x=0$.

$f(x)$ is continuous on $\mathbb{R} - \{0\}$.

25. $f(x) = \frac{\sin x}{\cos x}$

function is not defined on $x = (2n+1)\pi/2$ where 'n' is any integer.

Thus f is discontinuous at these points and continuous on all other points of $\mathbb{R}$.

$$26. \quad f(x) = \begin{cases} \sin x & \text{if } x \leq \pi/4 \\ \cos x & \text{if } x > \pi/4. \end{cases}$$

$$\text{Sol.} \quad \lim_{x \rightarrow \pi/4^-} f(x) = \lim_{x \rightarrow \pi/4^-} \sin x = \sin(\pi/4) = \frac{1}{\sqrt{2}}$$

$$\lim_{x \rightarrow \pi/4^+} f(x) = \lim_{x \rightarrow \pi/4^+} \cos x = \cos(\pi/4) = \frac{1}{\sqrt{2}}$$

$$f(\pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$$

Function is continuous at $x = \pi/4$. We also know that $\sin x$ and $\cos x$ are continuous at every point of $\mathbb{R}$. Hence $f(x)$ is continuous at every point of $\mathbb{R}$.

In Problem 27-34, examine whether the given function is continuous at $x=0$.

$$27. \quad f(x) = \begin{cases} (1+3x)^{1/x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0. \end{cases}$$

$$\text{Sol.} \quad \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (1+3x)^{1/x} = \lim_{x \rightarrow 0} [(1+3x)^{1/3x}]^3 = \left[\lim_{x \rightarrow 0} (1+3x)^{1/3x} \right]^3$$

$$\lim_{x \rightarrow 0} f(x) = e^3$$

$$f(0) = e^2$$

$$\lim_{x \rightarrow 0} f(x) \neq f(0) \quad \text{Function is not continuous.}$$

$$28. \quad f(x) = \begin{cases} (1+x)^{1/x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (1+x)^{1/x} = e$$

$$f(0) = 1$$

$$\lim_{x \rightarrow 0} f(x) \neq f(0)$$

Function is not continuous.

$$29. \quad f(x) = \begin{cases} (1+2x)^{1/x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases}$$

$$\text{Sol.} \quad \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (1+2x)^{1/x} = \lim_{x \rightarrow 0} [(1+2x)^{1/2x}]^2 = \left[\lim_{x \rightarrow 0} (1+2x)^{1/2x} \right]^2 = e^2$$

$$f(0) = e^2$$

So $\lim_{x \rightarrow 0} f(x) = f(0)$

So given $f(x)$ is continuous at $x=0$.

30.
$$f(x) = \begin{cases} e^{-1/x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Sol. $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} e^{-1/x} = e^{-1/0} = e^{-\infty} = 0$

$$f(0) = 1$$

$\lim_{x \rightarrow 0} f(x) \neq f(0)$ So given function is not continuous.

31.
$$f(x) = \begin{cases} \frac{e^{1/x}}{1+e^{1/x}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Sol. when $x \rightarrow 0^-$, $\frac{1}{x} \rightarrow -\infty$ and so $e^{1/x} \rightarrow e^{-\infty} = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{e^{1/x}}{1+e^{1/x}} = \frac{0}{1+0} = 0$$

when $x \rightarrow 0^+$, $\frac{1}{x} \rightarrow \infty$ and $e^{1/x} \rightarrow e^{\infty} = \infty$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{e^{1/x}}{1+e^{1/x}} = \lim_{x \rightarrow 0^+} \frac{e^{1/x}}{e^{1/x}(1 + \frac{1}{e^{1/x}})}$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{1 + \frac{1}{e^{1/x}}} = \frac{1}{1 + \frac{1}{\infty}}$$

$$\lim_{x \rightarrow 0^+} f(x) = \frac{1}{1+0} = 1$$

$\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$ function is not continuous at $x=0$.

32.
$$f(x) = \begin{cases} \frac{e^{1/x^2}}{e^{1/x^2} - 1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Sol. $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{e^{1/x^2}}{e^{1/x^2} - 1} = \lim_{x \rightarrow 0} \frac{e^{1/x^2}}{e^{1/x^2}(1 - \frac{1}{e^{1/x^2}})}$

$$= \lim_{x \rightarrow 0} \frac{1}{1 - \frac{1}{e^{1/x^2}}}$$

$$= \frac{1}{1 - \frac{1}{e^{\infty}}} = \frac{1}{1-0} = 1$$

$\lim_{x \rightarrow 0} f(x) = 1$ and $f(0) = 1$

$$\lim_{x \rightarrow 0} f(x) = f(0)$$

function is continuous at $x=0$.

$$33. \quad f(x) = \begin{cases} \frac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \quad \therefore \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Sol. $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2 \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} = 2 \cdot 1 = 2$
 $f(0) = 1$

$\lim_{x \rightarrow 0} f(x) \neq f(0)$
 function is discontinuous.

$$34. \quad f(x) = \begin{cases} \frac{\sin 3x}{\sin 2x} & \text{if } x \neq 0 \\ 2/3 & \text{if } x = 0 \end{cases}$$

Sol. $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 2x} = \frac{3}{2} \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot \frac{2x}{\sin 2x}$
 $= \frac{3}{2} \cdot \frac{\lim_{x \rightarrow 0} \frac{\sin 3x}{3x}}{\lim_{x \rightarrow 0} \frac{\sin 2x}{2x}} = \frac{3 \cdot 1}{2 \cdot 1}$

$$\lim_{x \rightarrow 0} f(x) = \frac{3}{2}$$

$$f(0) = 2/3$$

$\lim_{x \rightarrow 0} f(x) \neq f(0)$ function is not continuous at $x=0$

$$35. \text{ Let } f(x) = x^2 \text{ and } g(x) = \begin{cases} -4 & \text{if } x \leq 0 \\ |x-4| & \text{if } x > 0 \end{cases}$$

Determine whether fog and gof are continuous at $x=0$.

Sol. $f \circ g(x) = f\{g(x)\} = f(-4) \quad \text{if } x \leq 0$
 $f\{g(x)\} = f(|x-4|) \quad \text{if } x > 0$

Thus $(f \circ g)(x) = 16 \quad \text{if } x \leq 0$
 $(f \circ g)(x) = (x-4)^2 \quad \text{if } x > 0$

Now $\lim_{x \rightarrow 0^-} (f \circ g)(x) = 16$

$$\lim_{x \rightarrow 0^+} (f \circ g)(x) = \lim_{x \rightarrow 0^+} (x-4)^2 = (0-4)^2 = 16$$

$$(f \circ g)(0) = 16$$

Thus fog is continuous at $x=0$.

Again, $(g \circ f)(x) = g\{f(x)\} = g(x^2)$

$$(g \circ f)(x) = -4 \quad \text{if } x^2 \leq 0$$
$$= |x^2 - 4| \quad \text{if } x^2 > 0$$

$$\lim_{x \rightarrow 0^-} (g \circ f)(x) = -4$$

$$\lim_{x \rightarrow 0^+} (g \circ f)(x) = \lim_{x \rightarrow 0^+} |x^2 - 4|$$
$$= |0 - 4|$$

$$\lim_{x \rightarrow 0^+} (g \circ f)(x) = 4$$

$$\lim_{x \rightarrow 0^+} (g \circ f)(x) \neq \lim_{x \rightarrow 0^-} (g \circ f)(x)$$

Limit does not exist.

So $(g \circ f)(x)$ is not continuous at $x=0$.