

Therefore
$$\lim_{n \rightarrow \infty} \frac{\int_0^{\pi/2} \sin^{2n} x \, dx}{\int_0^{\pi/2} \sin^{2n+1} x \, dx} = 1, \quad (7)$$

by the Sandwiching Theorem 1.32 (v).

Taking limits of both sides of (5) as $n \rightarrow \infty$ we have

$$\begin{aligned} \frac{\pi}{2} &= \lim_{n \rightarrow \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2n)(2n)}{1 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots (2n-1)(2n+1)} \times \lim_{n \rightarrow \infty} \frac{\int_0^{\pi/2} \sin^{2n} x \, dx}{\int_0^{\pi/2} \sin^{2n+1} x \, dx} \\ &= \lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}, \text{ using (7).} \end{aligned}$$

This is known as Wallis' Product Formula for $\frac{\pi}{2}$.

Exercise Set 5.4

Evaluate (Problems 1 – 21):

1. $\int \frac{\sec^4 x}{\tan^5 x} \, dx$

2. $\int \sin^2 x \cos^4 x \, dx$

3. $\int \sin^6 x \cos^2 x \, dx$

4. $\int \sin^{1/2} x \cos^3 x \, dx$

5. $\int \sec^2 x \csc^3 x \, dx$

6. $\int \tan^3 x \sec^5 x \, dx$

7. $\int \cot^5 x \csc^4 x \, dx$

8. $\int \frac{\sin^2 x}{\cos^5 x} \, dx$

9. $\int_{\pi/4}^{\pi/2} \cot^4 x \, dx$

10. $\int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x \, dx$

11. $\int_0^{\pi/2} \tan^5 \left(\frac{x}{2} \right) dx$

12. $\int_0^a (a^2 - x^2)^{5/2} dx$

13. $\int_0^{\pi} \frac{\sin^4 x}{(1 + \cos x)^2} dx$

14. $\int_0^{\pi/4} \sin^4 2x \, dx$

15. $\int_0^{\pi/2} \sin^6 3x \, dx$

16. $\int_0^{\pi/8} \sin^5 4x \cos^4 4x \, dx$

17. $\int_0^{\pi/4} \cos^2 2x \, dx$

18. $\int_0^{\pi/6} \cos^3 3x \, dx$

19. $\int_0^{\pi/2} \sin^2 6x \cos^4 3x \, dx$

20. $\int_{\pi/6}^{\pi/2} \frac{\cos^2 x}{\sin x} \, dx$

21. $\int_0^1 \frac{x^6 \, dx}{\sqrt{1-x^2}}$

22. Show that

$$\int \sec^{2n+1} x \, dx = \frac{\sec^{2n-1} x \tan x}{2n} + \left(1 - \frac{1}{2n}\right) \int \sec^{2n-1} x \, dx.$$

23. Obtain a reduction formula for $\int \frac{dx}{(a^2 + x^2)^n}$, where n is an integer. Show

that $\int_0^{\infty} \frac{dx}{(1+x^2)^5} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \frac{\pi}{2}.$

24. If I_n denotes $\int_0^1 x^p (1-x^q)^n / dx$, where p, q and n are positive, prove that $(qn + p + 1) I_n = qn I_{n-1}$. Evaluate I_n when n is a positive integer.

25. Obtain a reduction formula for $\int \frac{x^n}{\sqrt{1-x^2}} dx$ and hence evaluate $\int \frac{x^3}{\sqrt{1-x^2}} dx.$

26. Calculate the value of $\int_0^{2a} x^n \sqrt{2ax - x^2} \, dx$, n being a positive integer.

Hence or otherwise calculate the values of

(i) $\int_0^{2a} x \sqrt{2ax - x^2} \, dx$

(ii) $\int_0^{2a} x^4 \sqrt{2ax - x^2} \, dx$

27. If $I_n = \int x^n (a^2 - x^2)^{1/2} dx$, prove that

$$I_n = -\frac{x^{n+1} (a^2 - x^2)^{3/2}}{n+2} + \frac{n-1}{n+2} a^2 I_{n-2}$$

Hence evaluate $\int_0^a x^4 \sqrt{a^2 - x^2} dx$.

28. Prove that

$$\int x^m (\ln x)^n dx = \frac{x^{m+1} (\ln x)^n}{m+1} - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx.$$

Hence calculate

$$(i) \int x^m (\ln x)^3 dx \qquad (ii) \int_0^1 x^m (\ln x)^n dx$$

29. Prove that

$$\int_0^{\pi/2} \cos^m x \sin nx dx = \frac{1}{m+n} + \frac{m}{m+n} \int_0^{\pi/2} \cos^{m-1} x \sin (n-1) x dx$$

Hence evaluate $\int_0^{\pi/2} \cos^6 x \sin 3x dx$.

30. Find a reduction formula for $\int \frac{x^n}{\sqrt{ax^2 + 2bx + c}} dx$.

Numerical Integration

In Chapter 4 we studied various methods of evaluating antiderivatives of certain functions. In order to evaluate definite integrals, the Fundamental Theorem of Integral Calculus (5.3) is a basic tool. But this theorem fails to deliver if the antiderivative of the integral cannot be found in terms of elementary functions (i.e., functions that can be expressed as a finite combination of algebraic and transcendental functions). For such cases Riemann sums provide an approximation of a definite integral when the number of points in partition is large. In practice this method is seldom used since there are better techniques and formulas which give a more efficient way to approximate such integrals. The methods of approximating definite integrals are called numerical integration. In this section, we discuss two such methods.

Ex #5.4

Q No. 1:- $\int \frac{\sec^4 x}{\tan^5 x} dx$

$\int \frac{\sec^4 x}{\tan^5 x} dx \rightarrow$ Put $\tan x = z$, $\sec x dx = dz$

$\int \frac{\sec^4 x}{\tan^5 x} dx = \int \frac{\sec^2 x}{\tan^5 x} \sec^2 x dx = \int \frac{1+z^2}{z^5} dz$

$= \int \left(\frac{1}{z^5} + \frac{1}{z^3} \right) dz = -\frac{1}{4z^4} - \frac{1}{2z^2}$

$\int \frac{\sec^4 x}{\tan^5 x} dx = -\frac{1}{4\tan^4 x} - \frac{1}{2\tan^2 x}$ Ans-

Q No. 2:- $\int \sin^2 x \cos^4 x dx$

We know that,

$\int \sin^p x \cos^q x dx = \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sin^p x \cos^{q-2} x dx$

Put $p=2$, $q=4$

$\rightarrow \int \sin^2 x \cos^4 x dx = \frac{\sin^3 x \cos^3 x}{6} + \frac{1}{2} \int \sin^2 x \cos^2 x dx$

Now

$\int \sin^2 x \cos^2 x dx = \frac{\sin^3 x \cos x}{4} + \frac{1}{4} \int \sin^2 x \cos^0 x dx$

$= \frac{\sin^3 x \cos x}{4} + \frac{1}{4} \int 1 - \cos 2x dx$

$= \frac{\sin^3 x \cos x}{4} + \frac{x}{8} - \frac{\sin 2x}{8 \cdot 2}$

We have,

$\int \sin^2 x \cos^4 x dx = \frac{\sin^3 x \cos^3 x}{6} + \frac{\sin^3 x \cos x}{8} + \frac{x}{16} - \frac{\sin x \cos x}{16}$

Ans

Q No. 3:- $\int \sin^6 x \cos^2 x dx$

We have reduction formula:-

$\int \sin^p x \cos^q x dx = -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x dx$

Put $P=6, q=2$ Then,

$$I = \int \sin^6 x \cos^2 x dx = -\frac{\sin^5 x \cos^3 x}{8} + \frac{5}{8} \int \sin^4 x \cos^2 x dx$$

$$\rightarrow \int \sin^4 x \cos^2 x dx = -\frac{1}{6} \sin^3 x \cos^3 x + \frac{1}{2} \int \sin^2 x \cos^2 x dx$$

$$\Rightarrow \int \sin^2 x \cos^2 x dx = -\frac{1}{4} \sin x \cos^3 x + \frac{1}{4} \int \cos^2 x dx$$

$$= -\frac{1}{4} \sin x \cos^3 x + \frac{1}{4} \int \frac{1 + \cos 2x}{2} dx$$

$$= -\frac{1}{4} \sin x \cos^3 x + \frac{1}{4} \left(\frac{x}{2} + \frac{\sin 2x}{4} \right)$$

Therefore:-

$$I = -\frac{1}{8} \sin^5 x \cos^3 x + \frac{5}{8} \left[-\frac{1}{6} \sin^3 x \cos^3 x + \frac{1}{2} \left(-\frac{1}{4} \sin x \cos^3 x \right) + \frac{1}{8} \left(\frac{x}{2} + \frac{\sin 2x}{4} \right) \right]$$

$$I = -\frac{1}{8} \sin^5 x \cos^3 x - \frac{5}{48} \sin^3 x \cos^3 x - \frac{5}{64} \sin x \cos^3 x + \frac{5}{64} \left(\frac{x}{2} + \frac{\sin 2x}{4} \right)$$

$$I = -\frac{\sin^5 x \cos^3 x}{8} - \frac{5 \sin^3 x \cos^3 x}{48} - \frac{5 \sin x \cos^3 x}{64} + \frac{5x}{128} + \frac{5 \sin x \cos x}{128}$$

Ans

Q No. 4:- $\int \sin^{1/2} x \cos^3 x dx$

$$I = \int \sqrt{\sin x} \cos^3 x dx$$

$$\text{Put } \sqrt{\sin x} = z \Rightarrow z^2 = \sin x$$

$$2z dz = \cos x dx$$

$$I = \int z \cdot (1 - z^4) \cdot 2z dz$$

$$= 2 \int z^2 (1 - z^4) dz = 2 \int (z^2 - z^6) dz$$

$$I = \frac{2}{3} z^3 - \frac{2}{7} z^7$$

$$I = \frac{2}{3} \sin^{3/2} x - \frac{2}{7} \sin^{7/2} x$$

Ans

$$QNO.5:- \int \sec^2 x \csc^3 x dx$$

$$I = \int \sec^2 x \csc^3 x dx$$

Integration by Parts:-

$$I = \int \csc^3 x \cdot \sec^2 x dx$$

$$= \csc^3 x \cdot \tan x - \int 3 \csc^2 x \cdot (-\csc x \cot x) \cdot \tan x dx$$

$$I = \csc^3 x \tan x - \int -3 \csc^3 x \cot x \tan x dx$$

$$I = \csc^3 x \tan x + 3 \int \csc^3 x dx$$

Now,

By using reduction formula:-

$$\int \csc^3 x dx = -\frac{1}{2} \csc x \cot x + \frac{1}{2} \int \csc x dx$$

$$= -\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln |\csc x - \cot x|$$

I becomes:-

$$I = \csc^3 x \tan x + 3 \left[-\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln |\csc x - \cot x| \right]$$

$$I = \csc^3 x \tan x - \frac{3}{2} \csc x \cot x + \frac{3}{2} \ln |\csc x - \cot x|$$

Ans

$$QNO.6:- \int \tan^3 x \sec^5 x dx$$

$$I = \int \tan^3 x \sec^5 x dx$$

$$\text{Put } \sec x = z, \quad \sec x \tan x dx = dz$$

$$I = \int \tan^2 x \sec^4 x (\sec x \tan x) dx$$

$$I = \int (z^2 - 1) z^4 dz = \int (z^6 - z^4) dz$$

$$I = \frac{z^7}{7} - \frac{z^5}{5} = \frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x \quad \text{Ans}$$

$$QNO.7:- \int \cot^5 x \csc^4 x dx$$

$$I = -\int \cot^5 x \csc^4 x (-\csc^2 x) dx$$

$$\text{Put } \cot x = z, \quad \csc^2 x dx = -dz$$

$$I = -\int z^5 (1+z^2) dz = -\int (z^5 + z^7) dz$$

$$I = -\frac{z^6}{6} - \frac{z^8}{8} = -\frac{1}{6} \cot^6 x - \frac{1}{8} \cot^8 x$$

Q No. 8: - $\int \frac{\sin^4 x}{\cos^5 x} dx$

$$I = \int \frac{\sin^2 x}{\cos^5 x} dx = \int \frac{\sin^2 x}{\cos^3 x} \cdot \frac{1}{\cos^2 x} dx$$

$$I = \int \tan^2 x \sec^3 x dx = \int (\sec^2 x - 1) \sec^3 x dx$$

$$I = \int \sec^5 x dx = \int \sec^3 x dx$$

Now, By reduction formula

$$\int \sec^5 x dx = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} \int \sec^3 x dx$$

So,

$$I = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} \int \sec^3 x dx = \int \sec^3 x dx$$

$$= \frac{\sec^3 x \tan x}{4} - \frac{1}{4} \int \sec^3 x dx$$

$$\text{Again, } \int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x dx$$

$$= \frac{\sec x \tan x}{2} + \frac{1}{2} \ln |\sec x + \tan x|$$

$$I = \frac{\sec^3 x \tan x}{4} - \frac{1}{8} \sec x \tan x - \frac{1}{8} \ln |\sec x + \tan x| \quad \text{Ans}$$

Q No. 9: - $\int_{\pi/4}^{\pi/2} \cot^4 x dx$

By reduction formula:-

$$\int \cot^4 x dx = -\cot^3 x - \int \cot^2 x dx$$

$$= -\cot^3 x - \int (\csc^2 x - 1) dx$$

$$= -\cot^3 x - (-\cot x) + x$$

$$= -\cot^3 x + \cot x + x$$

$$I = \int_{\pi/4}^{\pi/2} \cot^4 x dx = \left[-\cot^3 x + \cot x + x \right]_{\pi/4}^{\pi/2}$$

$$I = \frac{\pi}{2} - \left[-\frac{1}{3} + 1 + \frac{\pi}{4} \right] = \frac{\pi}{4} - \frac{2}{3} \quad \text{Ans}$$

Q No. 10:-

$$\int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x dx$$

Put $\csc x = z$, $-\csc x \cot x dx = dz$

$$I = \int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x dx = - \int_{\pi/4}^{\pi/2} (\csc^2 x - 1) \csc^2 x (\csc x \cot x) dx$$

$$I = - \int_{\pi/4}^{\pi/2} \csc^4 x - \csc^2 x (-\csc x \cot x) dx$$

$$= - \int_{\sqrt{2}}^1 (z^4 - z^2) dz = - \int_1^{\sqrt{2}} z^4 dz - \int_1^{\sqrt{2}} z^2 dz$$

$$I = \left[\frac{z^5}{5} - \frac{z^3}{3} \right]_1^{\sqrt{2}} = \left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} \right) - \left(\frac{1}{5} - \frac{1}{3} \right)$$

$$= \sqrt{2} \left(\frac{4}{5} - \frac{2}{3} \right) - \left(-\frac{2}{15} \right)$$

$$I = \frac{2}{15} (\sqrt{2} + 1) \text{ Ans}$$

Q No. 11:-

$$\int_0^{\pi/2} \tan^5 \left(\frac{x}{2} \right) dx$$

Put $\frac{x}{2} = z$, $dx = 2 dz$

When $x \rightarrow 0$, $z \rightarrow 0$
 $x \rightarrow \pi/2$, $z \rightarrow \pi/4$

$$I = \int_0^{\pi/2} \tan^5 \left(\frac{x}{2} \right) dx = 2 \int_0^{\pi/4} \tan^5 z dz$$

Now, $\int \tan^5 z dz = \tan^4 z \int \tan^3 z dz$

$$= \tan^4 z \left[\frac{\tan^2 z}{2} - \int \tan z dz \right]$$

$$= \frac{\tan^6 z}{2} - \frac{\tan^4 z}{2} + \ln |\sec z|$$

So,

$$I = 2 \left[\frac{\tan^6 z}{2} - \frac{\tan^4 z}{2} + \ln |\sec z| \right]_0^{\pi/4}$$

$$I = 2 \left[\left(\frac{1}{4} - \frac{1}{2} + \ln \sqrt{2} \right) - 0 \right]$$

$$I = 2 \left(-\frac{1}{4} \right) + 2 \log \sqrt{2} = -\frac{1}{2} + \ln 2 \text{ Ans}$$

Q No. 12:-

$$\int_0^{\pi} \frac{\sin^4 x}{(1 + \cos x)^2} dx$$

$$I = \int_0^{\pi} \frac{\sin^4 x}{(1 + \cos x)^2} dx = \int_0^{\pi} \frac{(1 - \cos^2 x)^2}{(1 + \cos x)^2} dx$$

$$I = \int_0^{\pi} \frac{(1 - \cos x)^2 (1 + \cos x)^2}{(1 + \cos x)^2} dx$$

$$I = \int_0^{\pi} (1 - \cos x)^2 dx = \int_0^{\pi} (1 - 2\cos x + \cos^2 x) dx$$

$$I = \int_0^{\pi} (1 - 2\cos x + \frac{1 + \cos 2x}{2}) dx$$

$$I = \left[x - 2\sin x + \frac{x}{2} + \frac{\sin 2x}{4} \right]_0^{\pi}$$

$$I = \left[\frac{3}{2}x - 2\sin x + \frac{\sin 2x}{4} \right]_0^{\pi}$$

$$I = \frac{3}{2}\pi - 0 = \frac{3\pi}{2} \text{ Ans}$$

Q No. 14:-

$$\int_0^a (a^2 - x^2)^{5/2} dx$$

Put $x = a \sin \theta$, $dx = a \cos \theta d\theta$

$$I = \int_0^a (a^2 - x^2)^{5/2} dx$$

$$I = \int_0^{\pi/2} (a^2 - a^2 \sin^2 \theta)^{5/2} a \cos \theta d\theta$$

$$I = a^6 \int_0^{\pi/2} \cos^6 \theta d\theta$$

$$I = a^6 \cdot \frac{(6-1)(6-3)(6-5)}{6 \cdot (6-2)(6-4)} \cdot \frac{\pi}{2} \quad (\text{Wallis formula})$$

$$I = a^6 \cdot \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$I = \frac{5a^6 \pi}{32} \text{ Ans}$$

Q No. 13:- $\int_0^{\pi/4} \sin^4 x dx$

Put $2x = t$, $2dx = dt$

When $x \rightarrow 0$, $t \rightarrow 0$
 $x \rightarrow \frac{\pi}{4}$, $t \rightarrow \frac{\pi}{2}$

$$I = \frac{1}{2} \int_0^{\pi/2} \sin^4 t dt = \frac{1}{2} \cdot \frac{3 \cdot 1 \cdot \pi}{4 \cdot 2 \cdot 2} = \frac{3\pi}{32} \text{ Ans}$$

Q No. 15:- $\int_0^{\pi/6} \sin^6 3x dx$

Put $3x = t$, $3dx = dt$

When $x \rightarrow 0$, $t \rightarrow 0$
 $x \rightarrow \frac{\pi}{6}$, $t \rightarrow \frac{\pi}{2}$

$$I = \int_0^{\pi/6} \sin^6 3x \, dx = \int_0^{\pi/2} (\sin^6 t) \cdot \frac{1}{3} \, dt$$

$$I = \frac{1}{3} \int_0^{\pi/2} \sin^6 t \, dt$$

$$I = \frac{1}{3} \cdot \frac{(6-1)(6-3)(6-5)}{6 \cdot (6-2)(6-4)} \cdot \frac{\pi}{2}$$

$$I = \frac{1}{3} \cdot \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{5\pi}{96} \text{ Am}$$

QNo. 16:-

$$\int_0^{\pi/8} \sin^5 4x \cos^4 4x \, dx$$

Put $4x = z$, $dx = \frac{1}{4} dz$

when $x \rightarrow 0$, $z \rightarrow 0$

$x \rightarrow \frac{\pi}{8}$, $z \rightarrow \frac{\pi}{2}$

So, $I = \int_0^{\pi/8} \sin^5 4x \cos^4 4x \, dx = \int_0^{\pi/2} \sin^5 z \cos^4 z \cdot \frac{1}{4} \, dz$

$$I = \frac{1}{4} \int_0^{\pi/2} \sin^5 z \cos^4 z \, dz$$

Hence, P is odd and q is even. So,

$$\frac{1}{4} \int_0^{\pi/2} \sin^5 z \cos^4 z \, dz = \frac{1}{4} \left[\frac{(5-1)(5-3)(4-1)(4-3)}{(5+4)(5+4-2)(5+4-4)(5+4-6)(5+4-8)} \right]$$

$$\frac{\pi/4}{4} = \frac{1}{4} \left(\frac{4 \cdot 2 \cdot 3 \cdot 1}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} \right) = \frac{2}{315} \text{ Am}$$

QNo. 17:-

$$\int_0^{\pi/4} \cos^2 2x \, dx$$

Put $2x = t$, $dx = \frac{1}{2} dt$

when $x \rightarrow 0$, $t \rightarrow 0$

$x \rightarrow \frac{\pi}{4}$, $t \rightarrow \frac{\pi}{2}$

So, $I = \int_0^{\pi/4} \cos^2 2x \, dx = \int_0^{\pi/2} \cos^2 t \cdot \frac{1}{2} \, dt = \frac{1}{2} \int_0^{\pi/2} \cos^2 t \, dt$

$$I = \frac{1}{2} \left(\frac{2-1}{2} \cdot \frac{\pi}{2} \right) = \frac{\pi}{8} \text{ Am}$$

QNo. 18:-

$$\int_0^{\pi/6} \cos^3 3x \, dx$$

Put $3x = t$, $dx = \frac{1}{3} dt$

when $x \rightarrow 0$, $t \rightarrow 0$

$x \rightarrow \frac{\pi}{6}$, $t \rightarrow \frac{\pi}{2}$

$$I = \int_0^{\pi/6} \cos^3 3x \, dx = \int_0^{\pi/2} \cos^3 t \cdot \frac{1}{3} \, dt = \frac{1}{3} \int_0^{\pi/2} \cos^3 t \, dt$$

$$I = \frac{4}{3} \cdot \frac{(3-1)}{3} = \frac{2}{9} \text{ Ans.}$$

Q No. 19:-

$$I = \int_0^{\pi/3} \sin^2 3x \cos^4 3x dx$$

$$I = \int_0^{\pi/3} (2 \sin 3x \cos 3x) \cos^3 3x dx$$

$$= 4 \int_0^{\pi/3} \sin^2 3x \cos^6 3x dx$$

$$\rightarrow \text{Put } 3x = z, dx = \frac{1}{3} dz$$

$$\text{When } x \rightarrow 0, z \rightarrow 0$$

$$x \rightarrow \frac{\pi}{3}, z \rightarrow \pi$$

$$\text{So, } I = \frac{4}{3} \int_0^{\pi} \sin^2 z \cos^6 z dz$$

$$= \frac{8}{3} \int_0^{\pi/2} \sin^2 z \cos^6 z dz$$

$$\text{Where, } p=2, q=6$$

$$I = \frac{8}{3} \left(\frac{(2-1)(6-1)(6-3)(6-5)}{(2+6)(2+6-2)(2+6-4)(2+6-6)} \cdot \frac{\pi}{2} \right)$$

$$I = \frac{8}{3} \left(\frac{1 \cdot 5 \cdot 3 \cdot 1 \cdot \pi}{8 \cdot 6 \cdot 4 \cdot 2 \cdot 2} \right) = \frac{5\pi}{96} \text{ Ans}$$

Q No. 20:- $\int_{\pi/3}^{\pi/2} \frac{\cos^2 x}{\sin x} dx$

$$\text{Put } \cos x = z$$

$$dz = -\sin x dx = \frac{\sin x}{\sin x} \cdot -\sin x dx$$

$$dz = -\sin^2 x \cdot \frac{dx}{\sin x}$$

$$\frac{1}{1-z^2} dz = \frac{dx}{\sin x}$$

$$\text{When, } x \rightarrow \frac{\pi}{3}, z \rightarrow \frac{1}{2}$$

$$I = \int_{\pi/3}^{\pi/2} \frac{\cos^2 x}{\sin x} dx$$

$$x \rightarrow \frac{\pi}{2}, z \rightarrow 0$$

$$I = \int_{1/2}^0 z^2 \cdot \left(-\frac{1}{1-z^2} \right) dz = \int_0^{1/2} \frac{z^2}{1-z^2} dz$$

$$I = \int_0^{1/2} \left(1 - \frac{1}{1-z^2} \right) dz$$

$$1-z^2 \sqrt{\frac{z^2}{1-z^2}} = -1$$

$\frac{1}{2}$ By Partial fraction:-

$$I = \int_0^{\frac{1}{2}} \left[1 - \frac{1}{2} \left(\frac{1}{1+z} + \frac{1}{1-z} \right) \right] dz$$

$$I = \left[z \right]_0^{\frac{1}{2}} - \frac{1}{2} \left[\ln(1+z) \right]_0^{\frac{1}{2}}$$

$$I = \left(\frac{1}{2} - 0 \right) - \frac{1}{2} \left[\ln 1 + \ln \left(\frac{3}{2} \right) \right]$$

$$I = -\frac{1}{2} - \left(\frac{1}{2} \ln 3 \right) = -\frac{1}{2} + \ln \sqrt{3} \text{ Ans}$$

$$I = -\frac{1}{2} + \ln \sqrt{3} \text{ Ans}$$

Q No. 21:- $\int_0^1 \frac{x^6 dx}{\sqrt{1-x^2}}$

Put $x = \sin \theta$, $dx = \cos \theta d\theta$

When, $x \rightarrow 0$, $\theta \rightarrow 0$

$x \rightarrow 1$, $\theta \rightarrow \frac{\pi}{2}$

Therefore, $\int_0^1 \frac{x^6 dx}{\sqrt{1-x^2}} = \int_0^{\pi/2} \frac{\sin^6 \theta \cdot \cos \theta d\theta}{\cos \theta}$

$$= \int_0^{\pi/2} \sin^6 \theta d\theta$$

$$= \frac{(6-1)(6-3)(6-5) \cdot \pi}{6 \cdot (6-2)(6-4) \cdot 2}$$

$$= \frac{5 \cdot 3 \cdot 1 \cdot \pi}{6 \cdot 4 \cdot 2 \cdot 2}$$

$$= \frac{5\pi}{32} \text{ Ans}$$

Q No. 22:-

$$\int \sec^{2n+1} x dx = \frac{\sec x \tan x}{2n} + \left(1 - \frac{1}{2n}\right) \int \sec^{2n-1} x dx$$

$$I = \int \sec^{2n+1} x dx = \int \sec x \cdot \sec^{2n} x dx$$

$$I = \sec x \cdot \tan x - \int \tan x (2n-1) \sec x \cdot \sec^{2n-1} x dx$$

$$I = \sec x \cdot \tan x - (2n-1) \int \sec^{2n} x \cdot \tan^2 x dx$$

$$I = \sec x \cdot \tan x - (2n-1) \int \sec^{2n} x (\sec^2 x - 1) dx$$

$$I = \sec x \cdot \tan x - (2n-1) \int \sec^{2n+1} x dx + (2n-1) \int \sec^{2n} x dx$$

$$(2n-1) I + I = \sec x \cdot \tan x + (2n-1) \int \sec^{2n} x dx$$

$$I = \frac{\sec x \tan x}{2n} + \left(1 - \frac{1}{2n}\right) \int \sec^{2n} x dx$$

Hence Proved

Q No. 23:- Obtain reduction formula

$$\int \frac{dx}{(a^2+x^2)^n}, \text{ where } n \text{ is an Integer.}$$

Also, Show That: $\int_0^{\infty} \frac{dx}{(1+x^2)^5} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{\pi}{2}$

Sol:-

$$\int \frac{dx}{(a^2+x^2)^n} = \int (a^2+x^2)^{-n} \cdot 1 \, dx$$

$$= (a^2+x^2)^{-n} \cdot x - \int -n(a^2+x^2)^{-n-1} \cdot 2x \cdot x \, dx$$

$$= x(a^2+x^2)^{-n} + \int 2n x^2 (a^2+x^2)^{-n-1} \, dx$$

$$= x(a^2+x^2)^{-n} + 2n \int x^2 (a^2+x^2)^{-n-1} \, dx$$

$$= x(a^2+x^2)^{-n} + 2n \int (x^2 + a^2 - a^2) (a^2+x^2)^{-n-1} \, dx$$

$$= x(a^2+x^2)^{-n} + 2n \int (a^2+x^2)^{-n} \, dx - 2na^2 \int (a^2+x^2)^{-n-1} \, dx$$

$$\Rightarrow 2na^2 \int \frac{dx}{(a^2+x^2)^{n+1}} = x(a^2+x^2)^{-n} + (2n-1) \int (a^2+x^2)^{-n} \, dx$$

Replace n into $n-1$, we have,

$$2(n-1)a^2 \int \frac{dx}{(a^2+x^2)^n} = \frac{x}{(a^2+x^2)^{n-1}} + (2n-3) \int \frac{dx}{(a^2+x^2)^{n-1}}$$

$$\int \frac{dx}{(a^2+x^2)^n} = \frac{x}{2(n-1)a^2 (a^2+x^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} \int \frac{dx}{(a^2+x^2)^{n-1}}$$

Integrating b/w the limit 0 to ∞ .

$$\int_0^{\infty} \frac{dx}{(a^2+x^2)^n} = \frac{2n-3}{2a^2(n-1)} \int_0^{\infty} \frac{dx}{(a^2+x^2)^{n-1}}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(1+x^2)^5} = \frac{7}{2 \cdot 4} \int_0^{\infty} \frac{dx}{(1+x^2)^4}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(1+x^2)^4} = \frac{5}{2 \cdot 3} \int_0^{\infty} \frac{dx}{(1+x^2)^3}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(1+x^2)^3} = \frac{3}{2 \cdot 2} \int_0^{\infty} \frac{dx}{(1+x^2)^2}$$

$$\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{1}{2 \cdot 1} \int_0^{\infty} \frac{dx}{(1+x^2)} = \frac{1}{2 \cdot 1} [\tan^{-1} x]_0^{\infty}$$

$$= \frac{1}{2} \cdot \frac{\pi}{2}$$

We have,

$$\int_0^{\infty} \frac{dx}{(1+x^2)^5} = \frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{\pi}{2}$$

Hence, Proved

Q No. 24:-

$$I_n = \int_0^1 x^p (1-x^v)^n dx$$

$$\text{Prove :- } (an + p + 1) I_n = an I_{n-1}$$

$$\begin{aligned} I &= \int x^p (1-x^v)^n dx = (1-x^v)^n \frac{x^{p+1}}{p+1} - \int \frac{x^{p+1}}{p+1} \cdot n(1-x^v)^{n-1} \cdot -vx^{v-1} dx \\ &= \frac{x^{p+1} (1-x^v)^n}{p+1} + \frac{vn}{p+1} \int x^{p+v} (1-x^v)^{n-1} dx \\ &= \frac{x^{p+1} (1-x^v)^n}{p+1} + \frac{vn}{p+1} \int x^p (1-x^v)^{n-1} dx \\ &= \frac{x^{p+1} (1-x^v)^n}{p+1} + \frac{vn}{p+1} \int x^p (1-x^v)^{n-1} dx \\ &= \frac{x^{p+1} (1-x^v)^n}{p+1} + \frac{vn}{p+1} \int x^p (1-x^v)^{n-1} dx \\ &= \frac{x^{p+1} (1-x^v)^n}{p+1} + \frac{vn}{p+1} \int x^p (1-x^v)^{n-1} dx \end{aligned}$$

So,

$$\left(1 + \frac{vn}{p+1}\right) \int x^p (1-x^v)^n dx = \frac{x^{p+1} (1-x^v)^n}{p+1} + \frac{vn}{p+1} \int x^p (1-x^v)^{n-1} dx$$

Now,

$$\begin{aligned} (an + p + 1) \int_0^1 x^p (1-x^v)^n dx &= \frac{x^{p+1} (1-x^v)^n}{p+1} \Big|_0^1 + \frac{vn}{p+1} \int_0^1 x^p (1-x^v)^{n-1} dx \\ (an + p + 1) I_n &= an I_{n-1} \end{aligned}$$

Hence Proved.

Evaluate I_n : n is +ve integer:-

$$I_n = \frac{an}{(an + p + 1)} I_{n-1}$$

$$I_{n-1} = I_n \cdot \frac{a(n-1)}{a(n-1) + p + 1}$$

$$I_{n-2} = \frac{a(n-2)}{a(n-2)+p+1} I_{n-3}$$

$$I_{n-3} = \frac{a(n-3)}{a(n-3)+p+1} I_{n-4}$$

$$I_3 = \frac{3a}{3a+p+1} I_2$$

$$I_2 = \frac{2a}{2a+p+1} I_1$$

$$I_1 = \frac{a}{a+p+1} I_0$$

$$\text{Now, } I_0 = \int_0^1 x^p dx = \frac{x^{p+1}}{p+1} \Big|_0^1 = \frac{1}{p+1} - 0 = \frac{1}{p+1}$$

we get:-

$$I_n = \frac{a(n) \cdot a(n-1) \cdot a(n-2) \cdot a(n-3) \cdots 3a \cdot 2a \cdot 1a \cdot 1}{(a(n)+p+1)(a(n-1)+p+1)(a(n-2)+p+1) \cdots (a+p+1)(p+1)}$$

$$I_n = \frac{a^n \cdot n!}{(a(n)+p+1)(a(n-1)+p+1)(a(n-2)+p+1) \cdots (a+p+1)(p+1)}$$

which is required.

Q No. 25:-

Obtain a reduction formula for

$$\int \frac{x^n}{\sqrt{1-x^2}} dx$$

$$\begin{aligned} I &= \int \frac{x^n}{\sqrt{1-x^2}} dx = \int x^n (1-x^2)^{-\frac{1}{2}} \frac{(-2x) dx}{(-2x)} \\ &= -\frac{1}{2} \int x^{n-1} (1-x^2)^{-\frac{1}{2}} (-2x) dx \\ &= -\frac{1}{2} \left\{ \frac{x^n (1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \int (n-1) x^{n-2} (1-x^2)^{-\frac{1}{2}+1} dx \right\} \end{aligned}$$

$$\begin{aligned} I &= -\frac{1}{2} \left\{ \frac{x^n \sqrt{1-x^2}}{\frac{1}{2}} - \int (n-1) x^{n-2} \sqrt{1-x^2} dx \right\} \\ &= -x^n \sqrt{1-x^2} + (n-1) \int x^{n-2} (1-x^2)^{\frac{1}{2}} dx \\ &= -x^n \sqrt{1-x^2} + (n-1) \int x^{n-2} (1-x^2)^{-\frac{1}{2}} (1-x^2) dx \end{aligned}$$

$$\begin{aligned} I &= -x^n \sqrt{1-x^2} + (n-1) \int x^{n-2} (1-x^2)^{-\frac{1}{2}} - (n-1) \int \frac{x^n}{\sqrt{1-x^2}} dx \\ (1+(n-1)) I &= -x^n \sqrt{1-x^2} + (n-1) \int \frac{x^{n-2}}{\sqrt{1-x^2}} dx \end{aligned}$$

$$n I = -x^{n-1} \sqrt{1-x^2} + n-1 \int \frac{x^{n-2}}{\sqrt{1-x^2}} dx$$

$$I = -\frac{x^{n-1} \sqrt{1-x^2}}{n} + \frac{n-1}{n} \int \frac{x^{n-2}}{\sqrt{1-x^2}} dx$$

Evaluate:- $\int \frac{x^3}{\sqrt{1-x^2}} dx$

Put $n=3$ in above eq.

$$I = \int \frac{x^3}{\sqrt{1-x^2}} dx = -\frac{x^2 \sqrt{1-x^2}}{3} + \frac{2}{3} \int \frac{x}{\sqrt{1-x^2}} dx$$

$$I = -\frac{x^2 \sqrt{1-x^2}}{3} + \frac{2}{3(-2)} \int \frac{-2x}{\sqrt{1-x^2}} dx$$

$$I = -\frac{x^2 \sqrt{1-x^2}}{3} + (-\frac{1}{3}) \sqrt{1-x^2}$$

$$I = -\frac{x^2 \sqrt{1-x^2}}{3} - \frac{1}{3} \sqrt{1-x^2}$$

Ans

2a Q No. 26:- Calculate value of $\int_0^a x^n \sqrt{2a-x} dx$, n is +ve integer.

Sol:-

$$I' \rightarrow \int x^n \sqrt{2a-x} dx = \int x^n \sqrt{x} \sqrt{2a-x} dx$$

$$\int x^n \sqrt{2a-x} dx = \int x^{n+\frac{1}{2}} \sqrt{2a-x} dx = \int x^{n+\frac{1}{2}} \sqrt{2a-x} (-1) dx$$

$$I' = - \left\{ \frac{x^{n+\frac{1}{2}+1}}{\frac{1}{2}+1} \sqrt{2a-x} - \int (n+\frac{1}{2}) x^{n+\frac{1}{2}-1} \cdot \frac{(2a-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} dx \right\}$$

$$I' = - \left\{ \frac{x^{n+\frac{3}{2}} (2a-x)^{3/2}}{3/2} - \frac{2n+1}{3} \int x^{n+\frac{1}{2}-1} (2a-x)^{1/2} (2a-x) dx \right\}$$

$$I' = -\frac{2}{3} x^{n+\frac{3}{2}} (2a-x)^{3/2} + \frac{2n+1}{3} \int 2ax (2a-x)^{1/2} - x (2a-x)^{3/2} dx$$

$$I' = -\frac{2}{3} x^{n+\frac{3}{2}} (2a-x)^{3/2} + \frac{2n+1}{3} \int (2ax (2a-x)^{1/2} - x (2a-x)^{3/2}) dx$$

$$(1 + \frac{2n+1}{3}) I' = -\frac{2}{3} x^{n+\frac{3}{2}} (2a-x)^{3/2} + \frac{2(2n+1)}{3} \int ax (2a-x)^{1/2} dx$$

$$(2n+4) I' = -2 x^{n+\frac{3}{2}} (2a-x)^{3/2} + 2(2n+1) \int ax (2a-x)^{1/2} dx$$

$$I' = -\frac{x^{n+\frac{3}{2}} (2a-x)^{3/2}}{(n+2)} + \frac{(2n+1)}{n+2} \int ax (2a-x)^{1/2} dx$$

Therefore,

$$I_n = a \cdot \frac{2n+1}{n+2} \int_0^{2a} x^{n+\frac{1}{2}-1} \sqrt{2a-x} dx$$

So,

$$I_n = \frac{2n+1}{n+2} a I_{n-1}$$

$$I_{n-1} = \frac{2(n-1)+1}{(n-1)+2} a I_{n-2} = \frac{(2n-1)a}{n+1} I_{n-2}$$

$$I_{n-2} = \frac{(2n-3)a}{n} I_{n-3}$$

$$I_3 = \frac{7a}{5} I_2; \quad I_2 = \frac{5a}{4} I_1$$

$$I_1 = \frac{3a}{2} I_0$$

Now, $I_0 = \int_0^{2a} x^{1/2} \sqrt{2a-x} dx$

$\rightarrow$ Put $x = 2a \sin^2 \theta; dx = 4a \sin \theta \cos \theta d\theta$

$$I_0 = \int_0^{\pi/2} \sqrt{2a-x} dx$$

$$I_0 = \int_0^{\pi/2} \sqrt{4a^2 \sin^2 \theta} = 4a^2 \sin^2 \theta \cdot (4a \sin \theta \cos \theta) d\theta$$

$$I_0 = \int_0^{\pi/2} \sqrt{4a^2 \sin^2 \theta} \cdot \cos^2 \theta \cdot 4a \sin \theta \cos \theta d\theta$$

$$I_0 = \int_0^{\pi/2} 2a \sin \theta \cdot \cos \theta \cdot 4a \sin \theta \cos \theta d\theta$$

$$I_0 = \int_0^{\pi/2} 8a^2 \sin^2 \theta \cos^2 \theta d\theta$$

$$I_0 = 8a^2 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta$$

$$I_0 = 8a^2 \cdot \frac{2 \cdot 1 \cdot \pi}{4 \cdot 2 \cdot 2} = \frac{a^2 \pi}{2}$$

So, we have:-

$$I_n = \frac{(2n+1)(2n-1) \dots 5 \cdot 3}{(n+2)(n+1)n \dots 4 \cdot 3} a^n \cdot \frac{a^2 \pi}{2}$$

$$I_n = \frac{(2n+1)(2n-1) \dots 5 \cdot 3}{(n+2)(n+1) \dots 4 \cdot 3} a^{n+2} \cdot \frac{\pi}{2} \quad (A)$$

i):- Evaluate:-

$$I_1 = \int_0^{2a} x \sqrt{2a-x^2} dx$$

There are two ways to solve this I_1 .

and also $I_1 = (1)$ Put, $x = 2a \sin^2 \theta d\theta$

$\Rightarrow$ (ii):- Put $n=1$ in eq (A):-

$$I_1 = \frac{3}{3} \cdot \frac{a^3 \pi}{2} = \frac{a^3 \pi}{2}$$

ii) :- $I_4 = \int_0^{2a} x^4 \sqrt{2ax - x^2} dx$

We use eq. (A).

Put $n=4$ in (A).

$$I_4 = \frac{9 \cdot 7 \cdot 5 \cdot 3}{6 \cdot 5 \cdot 4 \cdot 3} a^6 \cdot \frac{\pi}{2} = \frac{21a^6 \pi}{16} \text{ Ans}$$

Q No. 27 :- $I_n = \int x^n (a^2 - x^2)^{1/2} dx$
Prove that :-

$$I_n = \frac{x^{n-1} (a^2 - x^2)^{3/2}}{n+2} + \frac{n-1}{n+2} a^2 I_{n-2}$$

Sol :-

$$I_n = \int x^n (a^2 - x^2)^{1/2} dx = -\frac{1}{2} \int x^{n-1} (a^2 - x^2)^{1/2} (-2x) dx$$

$$= -\frac{1}{2} \left\{ x^{n-1} (a^2 - x^2)^{3/2} - \int (n-1) x^{n-2} (a^2 - x^2)^{3/2} dx \right\}$$

$$I_n = -\frac{1}{2} \left\{ \frac{2}{3} x^{n-1} (a^2 - x^2)^{3/2} - \int 2(n-1) x^{n-2} (a^2 - x^2)^{3/2} dx \right\}$$

$$I_n = -\frac{1}{3} x^{n-1} (a^2 - x^2)^{3/2} + \frac{(n-1)}{3} \int x^{n-2} (a^2 - x^2)^{1/2} (a^2 - x^2) dx$$

$$I_n = -\frac{1}{3} x^{n-1} (a^2 - x^2)^{3/2} + \frac{(n-1)}{3} \int a^2 x^{n-2} (a^2 - x^2)^{1/2} - x^n \sqrt{a^2 - x^2} dx$$

$$(1 + \frac{n-1}{3}) I_n = -\frac{x^{n-1} (a^2 - x^2)^{3/2}}{3} + \frac{(n-1)a^2}{3} \int x^{n-2} \sqrt{a^2 - x^2} dx$$

$$\Rightarrow I_n = -\frac{x^{n-1} (a^2 - x^2)^{3/2}}{n+2} + \frac{(n-1)}{n+2} a^2 I_{n-2}$$

Hence, Proved

Evaluate :- $\int_0^a x^4 \sqrt{a^2 - x^2} dx$

$$I_4 = \int_0^a x^4 \sqrt{a^2 - x^2} dx$$

$$\Rightarrow \int x^4 \sqrt{a^2 - x^2} dx = -\frac{x^3 (a^2 - x^2)^{3/2}}{6} + \frac{3}{6} a^2 I_{4-2}$$

Now,

$$\int x^2 \sqrt{a^2 - x^2} dx = -\frac{x (a^2 - x^2)^{3/2}}{4} + \frac{1}{4} a^2 I_0$$

So,

$$I_4 = \int_0^a x^4 \sqrt{a^2 - x^2} dx = \frac{1}{2} a^2 \cdot \frac{1}{4} a^2 \int_0^a x^0 (a^2 - x^2)^{1/2} dx$$

$$I_4 = \frac{a^4}{8} \int_0^a \sqrt{a^2 - x^2} dx$$

$$\Rightarrow I_4 = \frac{a^4}{8} \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$I_4 = \frac{a^4}{8} \cdot \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{\pi a^6}{32} \text{ Ans}$$

Q No. 28:-

$$\int x^m (\ln x)^n dx = \frac{x^{m+1} (\ln x)^n}{m+1} - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx$$

Sol:-

$$\begin{aligned} \int x^m (\ln x)^n dx &= (\ln x)^n \cdot \frac{x^{m+1}}{m+1} - \int n (\ln x)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^{m+1}}{m+1} dx \\ &= \frac{(\ln x)^n x^{m+1}}{m+1} - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx \quad (A) \\ &= (\ln x)^n x^{m+1} - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx \end{aligned}$$

which is required.

i):- $\int x^m (\ln x)^3 dx$

$$\int x^m (\ln x)^3 dx = \frac{x^{m+1} (\ln x)^3}{m+1} - \frac{3}{m+1} \int x^m (\ln x)^2 dx$$

$$\Rightarrow \int x^m (\ln x)^2 dx = \frac{x^{m+1} (\ln x)^2}{m+1} - \frac{2}{m+1} \int x^m (\ln x) dx$$

$$\begin{aligned} \Rightarrow \int x^m (\ln x) dx &= \frac{x^{m+1} (\ln x)}{m+1} - \frac{1}{m+1} \int x^m dx \\ &= \frac{x^{m+1} (\ln x)}{m+1} - \frac{x^{m+1}}{(m+1)^2} \end{aligned}$$

we have,

$$\begin{aligned} \int x^m (\ln x)^3 dx &= \frac{x^{m+1} (\ln x)^3}{m+1} - \frac{3x^{m+1} (\ln x)^2}{(m+1)^2} + \frac{6x^{m+1} \ln x}{(m+1)^3} \\ &\quad - \frac{6x^{m+1}}{(m+1)^4} \text{ Ans} \end{aligned}$$

ii):- $\int_0^1 x^m (\ln x)^n dx$

$$\begin{aligned} (A) \Rightarrow \text{we have } \int_0^1 x^m (\ln x)^n dx &= -\frac{n}{m+1} \int_0^1 x^m (\ln x)^{n-1} dx \\ &= -\frac{n}{m+1} I_{m,n-1} \end{aligned}$$

So, $I_{m,n} = -\frac{n}{m+1} I_{m,n-1}$

$$I_{m,n-1} = -\frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,2} = -\frac{2}{m+1} I_{m,1}$$

$$I_{m,1} = -\frac{1}{m+1} I_{m,0} = -\frac{1}{m+1} \cdot \frac{1}{m+1}$$

Therefore,

$$I_{m,n} = \frac{(-1)^n n!}{(m+n)!} \text{ Ans}$$

Q NO. 29:- $\int \cos^m x \sin nx \, dx$

$$I = \int \cos^m x \sin nx \, dx = \cos^m x \left(-\frac{\cos nx}{n} \right) - \int m \cos^{m-1} x \sin x \left(-\frac{\cos nx}{n} \right) dx$$

$$I = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} \int \cos^{m-1} x \sin x \cos nx \, dx$$

$$\text{Since, } \sin(n-1)x = \sin(nx-x) = \sin nx \cos x - \cos nx \sin x$$

$$\Rightarrow \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$$

Therefore:-

$$I = \int \cos^m x \sin nx \, dx = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} \int \cos^{m-1} x (\sin nx \cos x - \sin(n-1)x) \, dx$$

$$(1 + \frac{m}{n}) I = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} \int \cos^{m-1} x \sin(n-1)x \, dx$$

$$(m+n)I = -\cos^m x \cos nx + m \int \cos^{m-1} x \sin(n-1)x \, dx$$

$$I = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} \int \cos^{m-1} x \sin(n-1)x \, dx$$

Ans

Now

$$\int_0^{\pi/2} \cos^m x \sin nx \, dx = \frac{1}{m+n} + \frac{m}{m+n} \int_0^{\pi/2} \cos^{m-1} x \sin(n-1)x \, dx$$

Put $m=5, n=3$

$$\int_0^{\pi/2} \cos^5 x \sin 3x \, dx = \frac{1}{8} + \frac{5}{8} \int_0^{\pi/2} \cos^4 x \sin 2x \, dx$$

$$\Rightarrow \int_0^{\pi/2} \cos^4 x \sin 2x \, dx = \frac{1}{8} + \frac{4}{8} \int_0^{\pi/2} \cos^3 x \sin x \, dx$$

$$\Rightarrow \int_0^{\pi/2} \cos^3 x \sin x \, dx = \frac{1}{4} + \frac{3}{4} \int_0^{\pi/2} \cos^2 x \, dx$$



$$\begin{aligned} &= \frac{1}{4} + \frac{3}{4} \int_0^{\pi/2} (1 + \cos 2x) \, dx \\ &= \frac{1}{4} + \frac{3}{4} \left(\frac{x}{2} + \frac{\sin 2x}{4} \right) \Big|_0^{\pi/2} \\ &= \frac{1}{4} + \frac{3}{4} \left(\frac{\pi}{4} + 0 \right) \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_0^{\pi/2} \cos^3 x \sin x \, dx &= \frac{1}{4} + \frac{3}{4} \int_0^{\pi/2} \cos^2 x (\sin(1-1)x) \, dx \\ &= \frac{1}{4} + \frac{3}{4} \cdot 0 = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \text{So, } \int_0^{\pi/2} \cos^5 x \sin 3x \, dx &= \frac{1}{8} + \frac{5}{8} + \frac{20}{48 \cdot 4} = \frac{24+40+20}{192} = \frac{84}{192} \\ &= \frac{7}{16} \text{ Ans} \end{aligned}$$

Q No. 30:- Find a reduction formula for $\int \frac{x^n}{\sqrt{ax^2+2bx+c}} dx$

Sol:- $\int \frac{x^n}{\sqrt{ax^2+2bx+c}} dx \rightarrow I = \int x^n (ax^2+2bx+c)^{-\frac{1}{2}} dx$
 $= \int x^n (ax^2+2bx+c)^{-\frac{1}{2}} dx \cdot \frac{2an+2b-2c}{2an}$

Sol: $I = \frac{1}{2a} \int x^{n-1} (ax^2+2bx+c)^{-\frac{1}{2}} (2ax+2b-2c) dx$

$I = \frac{1}{2a} \int x^{n-1} (ax^2+2bx+c)^{-\frac{1}{2}} (2ax+2b-2c) dx - \frac{2b}{2a} \int x^{n-1} (ax^2+2bx+c)^{-\frac{1}{2}} dx$

By first Integral:-

$\rightarrow I_1 = \frac{1}{2a} \left\{ x^{n-1} (ax^2+2bx+c)^{\frac{1}{2}} \int (n-1)x^{n-2} \cdot (ax^2+2bx+c)^{-\frac{1}{2}} dx \right.$

$I_1 = \frac{x^{n-1} (ax^2+2bx+c)^{\frac{1}{2}}}{a} - \frac{n-1}{a} \int x^{n-2} \cdot (ax^2+2bx+c)^{-\frac{1}{2}+1} dx$

$I_1 = \frac{x^{n-1} (ax^2+2bx+c)^{\frac{1}{2}}}{a} - \frac{n-1}{a} \int x^{n-2} (ax^2+2bx+c)^{-\frac{1}{2}} \cdot ax^2 dx$

$\frac{n-1}{a} \cdot 2b \int x^{n-1} (ax^2+2bx+c)^{-\frac{1}{2}} dx = \frac{n-1}{a} \cdot c \int x^{n-2} (ax^2+2bx+c)^{-\frac{1}{2}} dx$

$I_1 = \frac{x^{n-1} \sqrt{ax^2+2bx+c}}{a} - \frac{n-1}{a} \int \frac{x^n}{\sqrt{ax^2+2bx+c}} dx - \frac{2(n-1)b}{a} \int \frac{x^{n-1}}{\sqrt{ax^2+2bx+c}} dx$

$-\frac{(n-1)c}{a} \int \frac{x^{n-2}}{\sqrt{ax^2+2bx+c}} dx$

We get:-

$I(1+n-1) = \frac{x^{n-1} \sqrt{ax^2+2bx+c}}{a} - \frac{(2n-2)b}{a} \int \frac{x^{n-1}}{\sqrt{ax^2+2bx+c}} dx - \frac{(n-1)c}{a}$

$\int \frac{x^{n-2}}{\sqrt{ax^2+2bx+c}} dx - \frac{b}{a} \int x^{n-1} (ax^2+2bx+c)^{-\frac{1}{2}} dx$

$I = \frac{x^{n-1} \sqrt{ax^2+2bx+c}}{a} - \frac{b(2n-1)}{an} \int \frac{x^{n-1}}{\sqrt{ax^2+2bx+c}} dx$

$-\frac{(n-1)c}{an} \int \frac{x^{n-2}}{\sqrt{ax^2+2bx+c}} dx$

Ans